Composite fermion wave functions as conformal field theory correlators

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It is known that a subset of fractional quantum Hall wave functions has been expressed as conformal field theory (CFT) correlators, notably the Laughlin wave function at filling factor $\nu = 1/m$ (m odd) and its quasiholes, and the Pfaffian wave function at $\nu = 1/2$ and its quasiholes. We develop a general scheme for constructing composite-fermion (CF) wave functions from conformal field theory. Quasiparticles at $\nu = 1/m$ are created by inserting anyonic vertex operators, $P_{\frac{1}{m}}(z)$, that replace a subset of the electron operators in the correlator. The one-quasiparticle wave function is identical to the corresponding CF wave function, and the two-quasiparticle wave function has correct fractional charge and statistics and is numerically almost identical to the corresponding CF wave function. We further show how to exactly represent the CF wavefunctions in the Jain series $\nu = s/(2sp+1)$ as the CFT correlators of a new type of fermionic vertex operators, $V_{p,n}(z)$, constructed from n free compactified bosons; these operators provide the CFT representation of composite fermions carrying $2p$ flux quanta in the n^{th} CF Landau level. We also construct the corresponding quasiparticle- and quasihole operators and argue that they have the expected fractional charge and statistics. For filling fractions 2/5 and 3/7 we show that the chiral CFTs that describe the bulk wave functions are identical to those given by Wen's general classification of quantum Hall states in terms of K-matrices and l- and t-vectors, and we propose that to be generally true. Our results suggest a general procedure for constructing quasiparticle wave functions for other fractional Hall states, as well as for constructing ground states at filling fractions not contained in the principal Jain series.

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I. INTRODUCTION

The evidence for an intriguing connection between conformal field theory (CFT) and the fractional quantum Hall effect (FQHE) was accumulating in the 1980s. It was realized that the effective low-energy theory of the FQHE is a topological field theory of the Chern-Simons type, where the exchange phases of the anyonic quasiparticles and quasiholes are coded in the braiding properties of the corresponding Wilson loops[1]. Witten's subsequent demonstration that the braiding of the Wilson loops are reflected in the correlation functions of certain CFTs[2] suggested a CFT-FQHE relationship, which was further strengthened by Wen, who proposed that the gapless chiral edge modes of a FQH-droplet are described by a chiral $1+1$ dimensional CFT[3]. It was also noticed that the holomorphic part of the Laughlin wave function takes the form of a correlator of bosonic exponents, or vertex operators, in a two dimensional CFT[4,5].

The 1991 paper by Moore and Read was particularly important since it synthesized many of these ideas and made an explicit conjecture about the CFT description of quantum Hall (QH) states containing two parts: 1. "Representative" electronic wave functions for the ground state and its quasiparticle and quasihole excitations are correlation functions, or, more precisely, conformal blocks, in a rational conformal field theory (RCFT) where the various particles correspond to different primary fields. 2. The very same RCFT describes the edge excitations of the corresponding FQH droplet. In their paper Moore and Read gave some striking circumstantial arguments to support their conjecture, and they also showed that many FQH states, namely the Laughlin state, the states in the Halperin-Haldane hierarchy, their quasihole excitations, the Halperin spin singlet state[6], and the Haldane-Rezayi spin singlet pairing state[7], may be represented in terms of conformal blocks. All this might have been criticized for being just a reformulation of old results, but Moore and Read also used the CFT formalism to propose a new $\nu = 1/2$ state, the so-called Pfaffian wave function, which is tentatively assigned to the observed $\nu = 5/2$ FQHE. The quasiholes in this state have charge $q = 1/4$ rather than $q = 1/2$ expected from the filling fraction, and exhibit non-Abelian fractional statistics. To establish the latter it was essential to use CFT technology.[44]

Despite this advance one and a half decades ago, the program of establishing a one-to-one correspondence between QH states and conformal field theory has remained incomplete. No explicit conformal field theory expressions have so far been established for many important FQHE states; in particular, despite interesting progress[10], this is the case for the ground state wave functions of the prominent FQHE series $\nu = s/(2sp \pm 1)$, and their related quasihole or quasiparticle excitations. (Expressions for the states in the Haldane-Halperin hierarchy were given in Ref. 4, but

these are indirect, involving multiple integrals over auxillary quasihole coordinates.) Surprisingly, a proper conformal field theory representation does not exist even for the quasiparticles – as opposed to quasiholes – of the FQHE state at $\nu = 1/m$ and the Pfaffian wave function at $\nu = 1/2$.[45]

It is worth reminding ourselves what we can hope to accomplish using CFT techniques: We cannot "derive" the FQHE wave functions, since the CFT does not contain any information about the actual interelectron interaction. It is true that the short distance behavior of the electronic wave functions is reflected in the operator product expansion of the pertinent CFT vertex operators, but only in the simplest cases can this be directly related to a potential of the Haldane-Kivelson-Trugman type. Thus we can only hope to get "representative wave functions" in the sense of Moore and Read, and any new candidate wave function suggested by the CFT approach must be tested and confirmed against exact solutions of the Schrödinger equation known for small systems. The crucial question is if the CFT wave functions are sufficiently natural and simple to give new insight into the physics of the problem, facilitate computations of quantities like local charge and braiding statistics, and most importantly, inspire new generalizations. Finally, we should point out that we know of no general microscopic principle that requires that the correlated quantum mechanical wave functions of interacting electrons in the lowest Landau level should be expressible as simple correlation functions of certain vertex operators in a two dimensional Euclidean rational conformal field theory.

An insight into the general FQHE states comes from the composite fermion (CF) formalism[12,13]. Here the experimentally prominent Jain states at $\nu = s/(2sp+1)$ are formed from s filled Landau levels of "composite fermions," which are electrons carrying $2p$ flux quanta. Other CF states, as *e.g.* the Pfaffian, which is the preferred candidate for the observed $\nu = 5/2$ state, can be formed by various BCS type pairing mechanisms[4,8]. In the CF description, a quasihole is obtained simply by removing a composite fermion from an incompressible FQHE state, and a quasiparticle is a composite fermion in a higher, otherwise empty CF Landau level (LL). (CF Landau levels are also called Λ levels.) Explicit wave functions are constructed for all ground states and their quasiparticle and quasihole excitations. (The asymmetry between quasiparticles and quasiholes occurs since they reside in different CF Landau levels.) The CF approach is very successful, both in comparison with experiments and with numerical studies of two-dimensional electron gases in strong magnetic fields[13].

The issue of fractional charge and fractional statistics of the composite fermions is a subtle one. The quasiparticles and quasiholes are composite fermions added to or removed from a CF Landau level. From one perspective, they have unit charge and fermionic statistics. Indeed, the addition of one composite fermion increases the number of electrons, and hence the net charge, by one unit, and the fermionic statistics of composite fermions has been confirmed by numerous experiments (*e.g.* the observation of their Fermi sea). On the other hand, the CF quasiparticles and quasiholes have a fractional "local charge" (where the local charge is the charge measured relative to the background FQHE state) and a fractional braiding statistics[13,14,15,16]. These properties capture the physics that adding or removing a composite fermion causes nonlocal changes in the state, because the vortex, a constituent of the composite fermion, is a nonlocal object. This should be contrasted with the analogous process in the integral QHE, which is essentially local (the Landau level projection destroys locality only on the scale of the magnetic length ℓ), and can be described by a local, charge-e operator $\psi^\dagger_\alpha(\vec{x})$, where the subscript denotes the Landau level index. No such local operator can be constructed for the creation of a composite fermion, since the local charge of the quasiparticle differs from that of the electron. The fractional statistics of the quasiparticles also implies that they cannot be described by local operators, as emphasized by Fröhlich and Marchetti[17]. Even though fractional charge and fractional statistics cannot be read off directly from the CF wave functions, they nonetheless contain that information, not surprising in view of the fact that the CF construction provides a good description of all the low energy states. We mention here the quasiparticles at $\nu = 1/m$, for which the CF wave function differs from that proposed earlier by Laughlin[18]. The calculation of the Berry phase associated with two-CF quasiparticle exchange, originally performed by Kjønsberg and Leinaas[15] and subsequently by Jeon and collaborators[16], shows that the braiding statistics for the CF quasiparticles has a sharply defined fractional value; for the Laughlin quasiparticles, in contrast, numerical calculations do not produce a convergent result for the statistical angle[19].

In this paper we establish a firm connection between CF wave functions and CFT correlators. Specifically:

1. We construct the quasiparticles of $\nu = 1/m$ (m odd) using a new kind of anyonic vertex operators $P_{\frac{1}{m}}$. For a single quasiparticle, the resulting wave function is identical to that obtained using the CF theory. A generalization to two or more quasiparticles produces wave functions that are very similar to the CF wave functions but not identical. For two quasiparticles at $\nu = 1/3$, the overlap between the two wave functions is typically 99.99% for as many as 40 electrons.

2. We show that the ground state wave functions in the Jain series $\nu = n/(2np+1)$ are exactly given by sums of CFT correlators of a set of vertex operators, V_{np}, which in the CF language correspond to creating composite fermions in higher CF Landau levels.

3. We generalize the construction of the quasiparticle operator $P_{\frac{1}{m}}$, as well as of the quasihole operators, to higher

levels in the Jain sequence; at level n, there are n independent hole operators and one quasiparticle operator. The vertex operator $V_{n,p}$ at level n is closely related to the quasiparticle operator at level $n-1$.

4. We demonstrate that the very CFT that yields the CF wave functions also directly defines an edge theory for the Jain states that is precisely the one expected from the general arguments given by Wen[3].

Our CFT construction has many advantages. (i) At the technical level, it produces accurate wave functions directly in the lowest Landau level with no need for projection, and the charge and statistics of the quasiparticles are revealed in the algebraic properties of the corresponding operators, just as in the case of the quasiholes of the $\nu = 1/m$ states. (ii) Although the effective edge theory for the Jain states was known from general principles, we provide a direct derivation from a CFT where the conformal blocks yield microscopically accurate bulk wave functions. (iii) It gives a new insight and suggests new extensions; a generalization of this work produces natural ansätze for quasiparticle wave functions for more complicated CF states such as the Moore-Read Pfaffian state, as well as for ground states at fractions (*e.g.*, 4/11), which do not belong to the principal Jain series.

The paper is organized as follows. In the next section we explain the basic ideas behind our construction and give explicit wave functions for one- and two-quasiparticles, as well as that for a quasiparticle-quasihole pair. The general structure of the CFT description of the states in the Jain series is discussed in section III, while the detailed technical proof for the equivalence between the CF and the CFT wave functions is left for Appendix B. In section IV we explain the construction of the edge theory, and in section V we construct localized quasiparticle states and show how to extract charge and statistics from the relevant Berry phases; the latter can be ascertained analytically if we make a random phase assumption. Some details of the calculations are found in Appendix C. Section V presents numerical calculations supporting our claims in sections II and V and, finally, a summary is found in section VII. A short report on parts of this work has been published previously[20].

II. ONE AND TWO QUASIPARTICLES IN THE LAUGHLIN STATE

A. The ground state and the quasihole states

We first review some of the basic formalism of the CFT construction of QHE wave functions, in particular the construction of the ground state and quasihole wave functions at the Laughlin fractions $\nu = 1/m$, where m is an odd integer. Following Moore and Read[4], we introduce the normal-ordered vertex operators,

$$V_1(z) \;=\; :e^{i\sqrt{m}\varphi_1(z)}: \tag{1}$$

$$H_{\frac{1}{m}}(\eta) \;=\; :e^{\frac{i}{\sqrt{m}}\varphi_1(\eta)}:, \tag{2}$$

where the normal ordering symbol $:\ :$, will be suppressed in the following. The free massless boson field, φ_1, is normalized so as to have the (holomorphic) two point function

$$\langle \varphi_1(z)\varphi_1(w)\rangle = -\ln(z-w)\,, \tag{3}$$

so that the the vertex operators obey the relations

$$\begin{aligned} e^{i\alpha\varphi_1(z)}e^{i\beta\varphi_1(w)} \;&=\; e^{i\pi\alpha\beta}e^{i\beta\varphi_1(w)}e^{i\alpha\varphi_1(z)} = (z-w)^{\alpha\beta}e^{i\alpha\varphi_1(z)+i\beta\varphi_1(w)} \\ &\sim\; (z-w)^{\alpha\beta}e^{i(\alpha+\beta)\varphi_1(w)} \end{aligned} \tag{4}$$

where the last line expresses the operator product expansion (OPE) in the limit $z \to w$. From (4) follows $V_1(z)V_1(w) + V_1(w)V_1(z) = 0$, and $H_{\frac{1}{m}}(z)H_{\frac{1}{m}}(w) - e^{i\pi/m}H_{\frac{1}{m}}(w)H_{\frac{1}{m}}(z) = 0$. The first of these reflects that the electrons are fermions, while the second is appropriate for fractional statistics as discussed in reference [4].

We normalize the (holomorphic) $U(1)$ charge density operator as

$$J(z) = \frac{i}{\sqrt{m}}\partial_z\varphi_1(z) \tag{5}$$

so the corresponding charge is given by

$$\mathcal{Q} = \frac{1}{\sqrt{m}}\frac{1}{2\pi}\oint dz\,\partial_z\varphi_1(z), \tag{6}$$

where the contour encircles the whole system. The $U(1)$ charges, $Q = 1$ of the electron and $Q = 1/m$ of the quasihole, can be read directly from the commutators $[\mathcal{Q}, V_1(z)] = V_1(z)$ and $[\mathcal{Q}, H_{\frac{1}{m}}(\eta)] = \frac{1}{m} H_{\frac{1}{m}}(\eta)$. It is noted that Q does not give the electric charge; rather it has the interpretation of vorticity as seen from (4). Introducing a positive vorticity in a homogenous state corresponds to a local depletion of the electron liquid, while a negative vorticity amounts to a local increase in density. Thus the excess electron number compared with the ground state created by an operator with $U(1)$ charge Q is given by

$$\Delta n = \delta n - Q, \tag{7}$$

where the integer δn is the number of electrons added by the operator. If the argument of the operator is an electron coordinate, z_i, one electron is added, while no electron is added if the argument is a quasihole coordinate η_i. (The idea of binding of an electron and m vortices was implicit in Laughlin's original work, and was made explicit by Halperin[6], Girvin and MacDonald[21] and Read[22].)

The total electric charge of a particle is given by $Q_{el} = -e\Delta n = e(Q - \delta n)$. Note that the excess charge associated with the addition of an electron is zero, as expected, because this expands the droplet without creating any local charge variation.

The (un-normalized) $\nu = 1/m$ Laughlin wave function can now be written as (for notational convenience, we write $\Psi(z_i)$ instead of $\Psi(\{z_i\})$):

$$\begin{aligned} \Psi_L(z_i) &= \langle 0|\mathcal{R}\{V_1(z_1)V_1(z_2)\dots V_1(z_{N-1})V_1(z_N)e^{-i\sqrt{m}\rho_m \int d^2z\, \varphi_1(z)}\}|0\rangle \\ &\equiv \langle V_1(z_1)V_1(z_2)\dots V_1(z_{N-1})V_1(z_N)\rangle_{1/m} \\ &= \prod_{i<j}(z_i - z_j)^m e^{-\sum_i |z_i|^2/4\ell^2}, \end{aligned} \tag{8}$$

where $\mathcal{R}$ denotes radial ordering. The second line defines the average $\langle \dots \rangle_{1/m}$, and the third follows for the ordering $|z_1| \geq |z_2| \geq \dots |z_N|$, which will be assumed below unless indicated otherwise. In the following, we shall suppress the subscript $1/m$ whenever it is clear to what ground state we are referring. The exponential operator in (8) corresponds to a constant background particle density, $\rho_m = -\rho_0/m$, where $\rho_0 = 1/2\pi\ell^2$ is the density of a filled Landau level. This is necessary since the $U(1)$ charge neutrality condition, known from the Coulomb gas formulation, in the CFT ensures that the correlator vanishes unless $N = \rho_m \int d^2z = \rho_m A$, which defines the area, A, of the system. As explained in reference [4], the background charge will produce the correct gaussian factor $e^{-\sum_i^N |z_i|^2/4\ell^2}$ characteristic of the lowest Landau level wave function. For a more detailed discussion of this background charge prescription, see Appendix A.

The wave function for a collection of Laughlin quasiholes is also easily written:

$$\Psi_L(\eta_1 \dots \eta_n; z_i) = \langle H_{\frac{1}{m}}(\eta_1)H_{\frac{1}{m}}(\eta_2)\dots H_{\frac{1}{m}}(\eta_n)V_1(z_1)V_1(z_2)\dots V_1(z_{N-1})V_1(z_N)\rangle. \tag{9}$$

In this case the charge neutrality condition reads $N + n/m = \rho_m A'$, indicating an expansion of the droplet. From the general relation (4) we get $H_{\frac{1}{m}}(z)V_1(w) + V_1(w)H_{\frac{1}{m}}(z) = 0$ which guarantees that (9) is uniquely defined and analytic in the electron coordinates.

Very little of the rather sophisticated mathematics of CFT will be used in this paper, but a few formal comments are in order. A CFT is in general not defined by a Lagrangian, but by an operator product algebra, or set of fusion rules, together with a specification of the field content defined by the so-called primary fields. The CFTs of interest here are defined by a Lagrangian describing a collection of free bosons, φ_i, compactified on circles of radius $R_i = \sqrt{m_i}$ where m_i are odd integers. The primary fields are given by the chiral vertex operators $V(z) = e^{i\sum_i \frac{q_i}{R_i}\varphi_i(z)}$ where the integers q_i define the charge lattice describing the possible "electric" charges in the Coulomb gas formulation of the CFT. The vertex operators satisfy an extended chiral algebra that, together with the charge lattice, defines the relevant CFT, which in this case is called a "rational torus" with radii $\sqrt{m_i}$; this is an example of a rational CFT. Acting on the primary fields with the generators of the conformal group gives families of "descendant fields", which can be expressed using derivatives of the parent primary fields. Such descendant fields will be important in the construction of quasiparticle operators presented in the next section. The full CFT contains fields of both chiralities and has correlation functions that can be written as (in general a sum over) products of holomorphic and anti-holomorphic factors, so-called conformal blocks. The holomorphic blocks are precisely the correlation functions of chiral vertex operators that we have identified with the electronic wave functions. In general, these blocks also depend parametrically on quasiparticle and quasihole coordinates, and acquire nontrivial phase factors, called monodromies, when these coordinates are transported along closed loops. It is these monodromies that reproduce the braiding phases that also can be calculated from the expectation values of Wilson loops in a Chern-Simons theory. A detailed discussion of the conditions that a CFT has to fulfill in order to describe a QH state can be found in Ref. 23.

B. One quasiparticle

The most immediate guess[4] for a quasiparticle operator would be to simply change the sign in the exponent in the quasihole operator of (2), *i.e.* to use $e^{-\frac{i}{\sqrt{m}}\varphi_1(\eta)}$. That, however, introduces unacceptable singular terms $\sim \prod_i (z_i-\eta)^{-1}$ in the electronic wave function. Inspired by the CF wave functions, we instead define a quasiparticle operator, $P_{\frac{1}{m}}(z)$, which has a $U(1)$ charge $(1-1/m)$, and that will *replace* one of the the original electron operators $V_1(z)$. We can thus think of $P(z)$ as a modified electron operator, but with a different amount of vorticity. The excess electric charge associated with such a modification is the difference between the charges of the operators V_1 and $P_{\frac{1}{m}}$ *i.e.* $\Delta Q_{el} = e((1-1/m)-1) = -e/m$, as appropriate for a quasiparticle at $\nu = 1/m$. The modified electron operator is given by

$$P_{\frac{1}{m}}(z) = \partial e^{i(\sqrt{m}-\frac{1}{\sqrt{m}})\varphi_1(z)}, \tag{10}$$

and the wave function for a single quasiparticle with angular momentum l is written as

$$\begin{aligned}\Psi^{(l)}_{1qp}(z_i) &= \mathcal{A}\{z_1^l e^{-|z_1|^2/4m\ell^2}\langle P_{\frac{1}{m}}(z_1)V_1(z_2).....V_1(z_N)\rangle\} \\ &= \sum_i (-1)^{i+1} z_i^l\, e^{-|z_i|^2/4m\ell^2}\langle P_{\frac{1}{m}}(z_i)\prod_{j\neq i} V_1(z_j)\rangle \\ &= \sum_i (-1)^i z_i^l \prod_{j<k}^{(i)}(z_j-z_k)^m \partial_i \prod_{l\neq i}(z_l-z_i)^{m-1},\end{aligned} \tag{11}$$

where $\mathcal{A}$ denotes anti-symmetrization of the coordinates. The second line follows by noting that the anti-symmetrized product has the form of a Slater determinant which is then expanded by the first row. From (4) we get $P_{\frac{1}{m}}(z)V_1(w) - V_1(w)P_{\frac{1}{m}}(z) = 0$, so the radial reordering of the quasiparticle operator does not give rise to any sign change. The anti-symmetrization with respect to the remaining coordinates is trivial since $V_1(z)V_1(w) + V_1(w)V_1(z) = 0$. The charge neutrality condition now reads $N - 1 + (1-\frac{1}{m}) = \rho_m A''$, so the droplet has undergone a small contraction, as expected for a quasiparticle.

While the exponent of (10) follows naturally from the above charge requirement (and may be viewed as a combination of an electron operator and an "inverse" quasihole operator), the derivative has been put in "by hand". Without the derivative, the wave function (11) can be shown to be identically zero. Technically, $P_{\frac{1}{m}}(z)$ is a descendant of the primary field, $e^{i(\sqrt{m}-\frac{1}{\sqrt{m}})\varphi(z)}$, a construction that naturally generalizes to more complicated QH states[24]. Note that the derivative in (11) acts only on the holomorphic part of the wave function.[46]

The quasiparticle wave function of (11) has a different character than those written above for the ground and the quasihole states, in that it is a sum over correlators, and that it involves prefactors $f_1(z_i) = z_i^l e^{-|z_i|^2/4m\ell^2}$. The factor z_i^l sets the angular momentum, while the exponential factor is chosen to give the correct lowest Landau level (LLL) electronic wave function: Due to its modified charge, the quasiparticle operator $P_{\frac{1}{m}}(z_i)$ gives rise to an exponential factor $\exp(-|z_i|^2(1-1/m)/4\ell^2)$, and the compensating prefactor ensures that the overall gaussian factor is $\exp\{-\sum_j |z_j|^2/(4\ell^2)\}$. Here and in the following, we suppress exponential factors of the correlators whenever convenient, but fully display all prefactors for clarity. It is suggestive that the prefactors f_1 precisely constitute the angular momentum l wave function $\psi_l(z) = z^l e^{-|z|^2/4m\ell^2}$ for a charge e/m particle in the LLL. Although we have no formal derivation of this, we find below a similar interpretation in the case of several quasiparticles, where their anyonic nature is also manifest.

As pointed out previously, the quasiparticle wave function above is obtained by modifying one of the electron operators, rather than inserting a new operator. This is very suggestive of the CF picture of a quasiparticle as an excitation of a composite fermion to a higher CF Landau level. In fact, what originally led us to construct the operator $P_{\frac{1}{m}}$ was the observation that the wave function (11) is *identical* to the corresponding CF wave function (Eq. 5 of ref. 25), which is known to have a good variational energy and the correct fractional charge. In spite of this identity, however, there are two differences between the present derivation and the CF construction that deserve to be noted: First, the present formalism is entirely within the lowest Landau level. The CF construction of wave functions, on the other hand, involves placing composite fermions in higher CF Landau levels and subsequently projecting onto the LLL by replacing all $\bar{z}$:s by derivatives in the resulting polynomial. Technically, of course, when deriving the one-quasiparticle wave function, the derivatives in (11) enter in the exact same places as those due to projection in the CF construction – but no projection is needed in the present formalism[26]. We return to this point in section IV, where we construct the ground states of the Jain sequences at $\nu = n/(2np+1)$.[47] Second, in spite of the close relation to composite fermions, the operator $P_{\frac{1}{m}}(z)$ is not fermionic, as can be seen from the commutation relation

$P_{\frac{1}{m}}(z)P_{\frac{1}{m}}(w) - e^{i\pi(m-2+1/m)}P_{\frac{1}{m}}(w)P_{\frac{1}{m}}(z) = 0$ or the OPE $P_{\frac{1}{m}}(z)P_{\frac{1}{m}}(w) \sim (z-w)^{m-4+\frac{1}{m}}\, e^{i\frac{2(m-1)}{\sqrt{m}}\varphi_1(w)}$, that follow from (4). The precise connection to composite fermions will be discussed in the section on the $\nu = 2/5$ state below. Although the fractional exponent $1/m$ suggests fractional statistics, one cannot directly read the statistical angle from the two-point function. This issue is discussed in more detail in section V.

C. Two or more quasiparticles

Based on the experience with the single quasiparticle case, we expect the wave function for M quasiparticles to be of the form

$$\Psi^{(l)}_{Mqp}(z_i) \;=\; \mathcal{A}\{f_M(z_1 \dots z_M)\langle P_{\frac{1}{m}}(z_1)\dots P_{\frac{1}{m}}(z_M)V_1(z_{M+1})\dots\dots V_1(z_N)\rangle\}. \tag{12}$$

The form of f_M is determined by the condition that the final electronic wave function be analytic and antisymmetric, with limiting behavior $\sim (z_p - z_q)^{m-1+l_{pq}}$, with the relative angular momenta $l_{pq} \geq 1$ and odd. Because the correlator gives non-analytic factors of the type $\partial_p\partial_q(z_p - z_q)^{m-2+1/m}$ from all contractions among quasiparticle operators, we choose

$$f_M(z_1 \dots z_M) = g(Z)\prod_{p<q}^{M}(z_p - z_q)^{1+l_{pq}-1/m}e^{-\sum_i^M |z_i|^2/4m\ell^2}\,,$$

where $Z = \frac{1}{N}\sum_{i=1}^{N} z_i$ is the center of mass coordinate. Again, the exponential factors are included to give the correct gaussian factor $\exp[-\sum_j |z_j|^2/(4\ell^2)]$ in the N-electron wave function. As anticipated in the case of one quasiparticle, f_M is just the LLL wave function of M anyons with fractional charge e/m.

To cast (12) in a form suitable for computation, we will use the following formula, which generalizes the expansion by a row used in (11) above:

$$\begin{aligned}\mathcal{A}\;\; &\{\prod_{p<q}^{M}(z_p - z_q)^{1-1/m+l_{pq}}P_{\frac{1}{m}}(z_1)\dots P_{\frac{1}{m}}(z_M)V_1(z_{M+1})\dots\dots V_1(z_N)\} \\ \sim\; &\sum_{\{i_n\}}(-1)^{\sum_{p=1}^{M} i_p}\mathcal{R}\{\prod_{p<q}^{M}(z_{i_p} - z_{i_q})^{1-1/m+l_{pq}}P_{\frac{1}{m}}(z_{i_1})\dots P_{\frac{1}{m}}(z_{i_M})V_1(z_{\bar{i}_{M+1}})\dots V_1(z_{\bar{i}_N})\},\end{aligned} \tag{13}$$

where the sum is over all subsets $\{i_1 \dots i_M\}$ of M of the N integers, and $\{\bar{i}_1 \dots \bar{i}_M\}$ is the conjugate subset of $N - M$ integers. The proof is found in Appendix B 1.

Using this result, the wave functions for two quasiparticles with total angular momentum L and relative angular momentum l can be written as

$$\Psi_{2qp}(z_i) = \;=\; \sum_{i<j}(-1)^{i+j}Z^L_{ij}(z_i - z_j)^{1+l-\frac{1}{m}}e^{-\frac{1}{4m\ell^2}(|z_i|^2+|z_j|^2)}\langle P_{\frac{1}{m}}(z_i)P_{\frac{1}{m}}(z_j)\prod_{k\neq i,j}V_1(z_k)\rangle, \tag{14}$$

where $Z_{ij} = (z_i + z_j)/2$. Evaluating the correlator we obtain the following explicit form for the wave function for two quasiparticles with relative angular momentum l and center of mass angular momentum L,

$$\begin{aligned}\Psi^{l,L}_{2qp}(z_i) \;&=\; \sum_{i<j}(-1)^{i+j}Z^L_{ij}(z_i - z_j)^{1+l-\frac{1}{m}}\partial_{z_i}\partial_{z_j}(z_i - z_j)^{m-2+\frac{1}{m}} \\ &\prod_{k}^{(ij)}(z_k - z_i)^{m-1}\prod_{l}^{(ij)}(z_l - z_j)^{m-1}\prod_{m<n}^{(ij)}(z_m - z_n)^m,\end{aligned} \tag{15}$$

where the derivatives act on the whole expression to their right, and $\prod_k{}^{(ij)} = \prod_{\substack{k=1\\k\neq i,j}}^{N}$ and $\prod_{i<j}^{(kl)} = \prod_{\substack{i<j\\i,j\neq k,l}}^{N}$.

The corresponding wave function in the CF approach is given by[25]

$$\begin{aligned}\tilde{\Psi}^{l,L}_{2qp}(z_i) \;&=\; \sum_{i<j}(-1)^{i+j}Z^L_{ij}(z_i - z_j)^{l}\partial_{z_i}\partial_{z_j}(z_i - z_j)^{m-1} \\ &\prod_{k}^{(ij)}(z_k - z_i)^{m-1}\prod_{l}^{(ij)}(z_l - z_j)^{m-1}\prod_{m<n}^{(ij)}(z_m - z_n)^m\,.\end{aligned} \tag{16}$$

The two wave functions differ by terms wherein the derivatives in (16) act on the factor $(z_i - z_j)^{1-\frac{1}{m}}$. It is known[25] that the CF wave function in (16) gives the correct fractional charge and statistics of the two-quasiparticle state. The first non-trivial test of our construction is therefore to check whether the CFT wave function (14) shares these good charge and statistics properties. This is indeed the case, as demonstrated by our numerical simulations, which are summarized in section V below. These results show that the two wave functions are essentially identical (for example, their overlap is 99.96% for 50 particles). This can be understood from the following heuristic arguments: First, since the derivatives in (16) act on a function which is a polynomial of order N in both z_i and z_j, this will generate $O(N^2)$ terms. It is unlikely that the few terms picked up by acting on the first factor will be significant. Secondly, these terms are sub-leading in the coordinate difference $(z_i - z_j)$ between the quasiparticles, and thus unlikely to affect qualitative properties.

D. Quasiparticles and quasiholes

Wave functions for pairs of quasiparticles and quasiholes can be constructed by inserting pairs of the corresponding operators into the CFT correlator for the Laughlin ground state. The simplest case is a quasiparticle at the origin together with a quasihole at position η, given by

$$\begin{aligned}\Psi_{qp-qh}(z_i,\eta) &= \mathcal{A}\{e^{-|z_1|^2/4m\ell^2}\langle P_{\frac{1}{m}}(z_1)V_1(z_2)\ldots\ldots V_1(z_N)\,H_{\frac{1}{m}}(\eta)\rangle\} \\ &= \sum_i(-1)^{i+1}\,e^{-|z_i|^2/4m\ell^2}\langle P_{\frac{1}{m}}(z_i)\prod_{j\neq i}V_1(z_j)\,H_{\frac{1}{m}}(\eta)\rangle \\ &= \sum_i(-1)^i\prod_{j<k}^{(i)}(z_j-z_k)^m\prod_{j\neq i}(z_j-\eta)\,\partial_i\prod_{l\neq i}(z_l-z_i)^{m-1}\,(z_i-\eta)^{1-\frac{1}{m}},\end{aligned} \tag{17}$$

where the antisymmetrization acts on the electron coordinates z_i only. More generally, a quasiparticle localized at some position η' away from the origin may be constructed as a coherent superposition of the angular momentum states given in (11).

For states with equally many quasiparticles and quasiholes, the background charge does not have to be changed from its ground state value. In this sense, wave functions of this type are the natural low energy bulk excitations that do not require any compensating edge charge. On a closed surface, no fractionally charged states are allowed.

III. COMPOSITE FERMION STATES IN THE JAIN SERIES

A. The $\nu = 2/5$ composite fermion ground state

In the composite fermion picture, the ground state wave functions at fillings $\nu = n/(2np+1)$ are constructed as n filled Landau levels of composite fermions with $2p$ flux quanta attached. In particular, the $\nu = 2/5$ state corresponds to filling the lowest two CF Landau levels. This state may thus be viewed as a "compact" state of $N/2$ quasiparticles, *i.e.* the CF:s in the second Landau level are in the lowest possible total angular momentum state.

To explore the connection to our CFT construction, we generalize the two-quasiparticle wave function (15) of the $1/m$ state to the M-quasiparticle case, with $M = N/2$, and consider a maximum density circular droplet obtained by putting all the quasiparticle pairs in their lowest allowed relative angular momentum ($\ell = 1$), and with zero angular momentum for the center of mass ($L = 0$). For simplicity we shall also take $m = 3$ (and suppress the subscript m on the operators) since the generalization to arbitrary odd m is obvious. Using (13) and evaluating the correlators, the wave function for M quasiparticles reads

$$\begin{aligned}\Psi_{Mqp}(z_i) &= \sum_{i_1<i_2<\cdots<i_M}(-1)^{\sum_k^M i_k}\prod_{k<l}^{M}(z_{i_k}-z_{i_l})^{\frac{5}{3}}\partial_{z_{i_1}}\partial_{z_{i_2}}\ldots\partial_{z_{i_M}}\prod_{k'<l'}^{M}(z_{i'_k}-z_{i'_l})^{\frac{4}{5}} \\ &\prod_{k_1}^{(i_2,i_3\ldots i_M)}(z_{k_1}-z_{i_1})^2\prod_{k_2}^{(i_1,i_3\ldots i_M)}(z_{k_2}-z_{i_2})^2\ldots\prod_{k_n}^{(i_1,i_2\ldots i_M)}(z_{k_M}-z_{i_M})^2\prod_{m<n}^{(i_1,i_2\ldots i_M)}(z_m-z_n)^3.\end{aligned} \tag{18}$$

Since the anyonic wave function on the first line has the form of a Jastrow factor, it is natural to introduce a second free bosonic field $\varphi_2(z)$. In fact, by defining

$$\tilde{V}(z) = e^{i\sqrt{\frac{5}{3}}\varphi_2(z)}\partial e^{i\frac{2}{\sqrt{3}}\varphi_1(z)}\,, \tag{19}$$

we find that (18) may be written in the following compact form

$$\Psi_{Mqp}(z_i) \;=\; \mathcal{A}\{\langle\prod_{i=1}^{M}\tilde{V}(z_i)\prod_{j=M+1}^{N}V_1(z_j)\rangle\} \tag{20}$$

i.e. as a sum of correlators of M $\tilde{V}$:s and $(N-M)$ V_1:s.

Again, this expression differs from the corresponding CF wave function only in the ordering of the derivatives and the Jastrow factors in the first line of (18). Indeed, as demonstrated in Appendix B, the CF wave function is obtained simply by moving all the derivatives all the way to the left. Let us therefore define

$$V_2(z) = \partial e^{i\frac{2}{\sqrt{3}}\varphi_1(z)}e^{i\sqrt{\frac{5}{3}}\varphi_2(z)}\,, \tag{21}$$

where the derivative now acts on both the exponentials, and consider the case $N = 2M$. We then find that the following sum of correlators of M V_2:s and M V_1:s:

$$\begin{aligned}\Psi^{\rm CF}_{2/5}(z_i) \;&=\; \mathcal{A}\{\langle\prod_{i=1}^{M}V_2(z_i)\prod_{j=M+1}^{2M}V_1(z_j)\rangle\} \\ &=\sum_{\substack{i_1<i_2\cdots i_M\\ \bar{i}_1<\bar{i}_2\ldots\bar{i}_M}}(-1)^{\sum_k^M i_k}\langle V_2(z_{i_1})\ldots V_2(z_{i_M})V_1(z_{\bar{i}_1})\ldots V_1(z_{\bar{i}_M})\rangle\end{aligned} \tag{22}$$

exactly reproduces the $(N = 2M)$-electron CF wavefunction for $\nu = 2/5$.

The operators $V_2(z_i)$, as opposed to the $P(z_i)$:s, are real fermionic operators in that they anticommute among themselves, but commute with $V_1(z_i)$:s, just as the $P(z_i)$:s. Note that the form of V_2 was determined entirely from the form of the maximum density M-quasiparticle wave function, so its fermionic nature was not an input. If we want to interpret V_2 as a composite *electron* operator, it should have the same charge as V_1. This is ensured if we redefine the charge density operator as

$$J(z) = \frac{i}{\sqrt{3}}\partial\varphi_1(z) + \frac{i}{\sqrt{15}}\partial\varphi_2(z). \tag{23}$$

This construction may seem *ad hoc* in the sense that we fix the coefficient of φ_2 by hand so as to obtain the correct charge. However, we shall see below that this choice is consistent, in that it produces the correct charge for the quasiholes in the $\nu = 2/5$ state.

Fulfillment of the charge neutrality condition for the vertex operators V_2 requires a background charge, which for the maximum density circular droplet can be assumed to be constant. Furthermore, this density must reproduce the correct exponential factor for electrons in the LLL. The latter is achieved by redefining the expectation value as

$$\langle\cdots\rangle_{2/5} \equiv \langle 0|\ldots e^{-i\sqrt{15}\tilde{\rho}_3\int_{\tilde{A}}d^2z\,\varphi_2(z)}e^{-i\sqrt{3}\rho_3\int_A d^2z\,\varphi_1(z)}|0\rangle, \tag{24}$$

where $\tilde{\rho}_3 = (1/15)\rho_0$, so the total background electron density is $(1/3 + 1/15)\rho_0 = (2/5)\rho_0$. We stress that this value is not an input, but follows from demanding that V_2 describe unit charge particles in the LLL, which was what led us to the above form (23) of the charge density operator. We now show that this state is indeed homogeneous, *i.e.* that the droplets formed by the $N/2 = M$ V_1:s and the M V_2:s have the same area. Charge neutrality gives the following conditions on the areas A and $\tilde{A}$ integrated over in (24),

$$\begin{aligned}\sqrt{3}M + \frac{2}{\sqrt{3}}M \;&=\; \sqrt{3}\rho_3 A \\ \sqrt{\frac{5}{3}}M \;&=\; \sqrt{15}\tilde{\rho}_3\tilde{A}\,,\end{aligned} \tag{25}$$

which implies $A = \tilde{A}$ and thus homogeneity. From the perspective of composite fermions, this correponds to two filled CF Landau levels, since the degeneracy is the same in all Landau levels. It would be interesting to redo the above construction on a closed manifold, where we would expect the concept of "filled CF Landau level" to emerge in a natural way from the condition that the correlators do not vanish.

Although it is possible to write general many-quasiparticle wave functions similar to the two particle wave function in (14), it is only the maximum density droplet of (18), and more generally the "compact" CF states[13], that allow for a simple expression in terms of conformal blocks as in (20); for general relative angular momenta one still has to explicitly put in compensating (anyonic) wave functions by hand. In this general case, there is also no reason for introducing a constant background charge different from that of the "parent" $\nu = 1/3$, so there is no natural way to obtain non-zero correlators even if we were to introduce the field $\varphi_2(z)$. As we see below, this would also be in conflict with the known properties of the charge 1/3 quasiholes.

B. The quasihole operators

To create quasiholes in the 2/5 state, the operator $H_{\frac{1}{3}}(\eta)$ of (2) is no longer appropriate since it does not give holomorphic electron wave functions, as is seen from, *e.g.* , $\langle V_2(z)H_{1/3}(\eta)\rangle \sim (z-\eta)^{2/3}$. Instead, it is necessary to include the second Bose field, φ_2, and construct quasihole operators of the form $H_{pq}(\eta) = e^{i\frac{p}{\sqrt{3}}\varphi_1(\eta)+i\frac{q}{\sqrt{15}}\varphi_2(\eta)}$. The coefficients p and q are determined from the requirements that (i) the wave function of any single quasihole be holomorphic, *i.e.* the power of the correlator between any quasihole operator and $V_1(z)$ or $V_2(z)$ be a non-negative integer, and (ii) the resulting hole operator *not* be expressible as a combination (product) of the other quasihole or vertex operators. These conditions uniquely determine the allowed coefficients p and q, and lead to the following two fundamental quasihole operators for the $\nu = 2/5$ state:

$$\begin{aligned} H_{01} &= e^{i\frac{3}{\sqrt{15}}\varphi_2(\eta)} \\ H_{10} &= e^{\frac{i}{\sqrt{3}}\varphi_1(\eta)-\frac{2i}{\sqrt{15}}\varphi_2(\eta)} \ . \end{aligned} \tag{26}$$

Using the charge operator corresponding to the charge density (23) one verifies that both these operators create quasiholes with charge 1/5. Note that this charge assignment is a prediction of our scheme, rather than an input, since the form of the charge operator (23) was determined independently from demanding V_2 to have unit charge. All other allowed vertex operators can be constructed as products of $H_{01}(\eta)$ and $H_{10}(\eta)$; the operators in (26) span the charge lattice.

It is an easy exercise to construct the explicit electron wave functions obtained by inserting the operators (26) in the correlator (22). Not surprisingly, a direct correspondence with the composite fermion picture is again revealed: Inserting the operator $H_{10}(\eta)$ (with $\eta = 0$ for simplicity) into the $\nu = 2/5$ ground state (22) exactly gives the wave function of a quasihole in the center of the lowest CF Landau level, while H_{01} gives a quasihole in the second CF Landau level. Taking the product of the two quasihole operators, one obtains a charge-2/5 operator which, in the CF language, reproduces the wave function of a vortex, *i.e.* (for $\eta = 0$) two quasiholes at the origin, one in each CF-Landau level.[13] Section V clarifies the relation between these quasihole operators and Wen's effective bulk and edge theories for the $\nu = 2/5$ quantum Hall state.

If we would attempt to use the operators V_1 and V_2 to describe a 1/3 state with a small number of quasiparticles (*e.g.* by putting a compensating charge at the edge or at infinity by hand), we would be forced to use the operators (26) for the quasiholes and thus be led either to a wrong charge assignment for the quasiholes or to redefine the charge operator as to make the V_2:s carry fractional charge. This again stresses that the form of the charge operator as well as the various vertex operators is intimately tied to the particular ground state under consideration.

C. The quasiparticle operator

The quasiparticle operator of the $\nu = 2/5$ state is constructed in the same spirit as $P_{\frac{1}{3}}$ given in (10), *i.e.* as a combination of an "inverse" quasihole operator and one of the electron operators, combined with an appropriate number of derivatives. Since in the 2/5 state there are two independent hole operators (H_{01} and H_{10} in (26)) and two electron operators (V_1 and V_2), it superficially looks as if as there are four quasiparticle candidates. However, it can be shown[24] that three of these are excluded as they do not produce non-zero wave functions, and one is left with

$$P_{2/5}(z) = \partial^2 e^{\frac{2i}{\sqrt{3}}\varphi_1(z)+\frac{2i}{\sqrt{15}}\varphi_2(z)} \tag{27}$$

which corresponds to combining H_{01} (a quasihole in the second CF Landau level) with V_2 (a composite fermion in the second CF Landau level). Again, the two derivatives are necessary in order to produce a non-zero wave function

$$\Psi_{1qp}(z_i) = \mathcal{A}\langle P_{2/5}(z_1)\prod_{i=2}^{M+1} V_2(z_i)\prod_{j=M+2}^{2M+1} V_1(z_j)\rangle, \tag{28}$$

and (28) is identical to the corresponding CF wave function. Note that, given the connection to composite fermions, it is very natural to have two different quasihole operators but only one quasiparticle operator: There are two filled CF LLs in which to create quasiholes, but the only way (except for higher excitations) to create a quasiparticle is to put one composite fermion in the third CF Landau level.

D. The $\nu = 3/7$ state and the Jain series

As a final explicit example, let us construct the ground state and quasiholes of the $\nu = 3/7$ state, *i.e.* the third level of the $\nu = s/(2s+1)$ Jain sequence. The generalization to the full Jain series is given in Appendix B 3 .

The 3/7 state is obtained from a correlator containing an equal number of V_1:s, V_2:s and the new operator V_3:

$$V_3(z) = P_{2/5}(z) e^{i\frac{7}{\sqrt{35}}\varphi_3(z)} = \partial^2 e^{i[\frac{2}{\sqrt{3}}\varphi_1(z)+\frac{2}{\sqrt{15}}\varphi_2(z)+\frac{7}{\sqrt{35}}\varphi_3(z)]} \tag{29}$$

and again, the result is precisely the $\nu = 3/7$ CF wave function (see appendix B 3). The relevant charge density operator, which ensures unit charge of V_3, is given by

$$J(z) = \frac{i}{\sqrt{3}}\partial\varphi_1(z) + \frac{i}{\sqrt{15}}\partial\varphi_2(z) + \frac{i}{\sqrt{35}}\partial\varphi_3(z). \tag{30}$$

It is easy to check that $V_3(z)$ is fermionic, but commutes with both V_1 and V_2, and that the wave function written in analogy with (22) has filling fraction $\nu = 3/7$. In the language of composite fermions, this corresponds to filling up three CF Landau levels. In analogy with the 2/5 state, one finds three independent charge-1/7 quasihole operators, which exactly correspond to quasiholes in the third, second, and first CF Landau levels, respectively:

$$\begin{aligned} H_{001}(\eta) &= e^{i\left[\frac{5}{\sqrt{35}}\varphi_3(\eta)\right]} \\ H_{010}(\eta) &= e^{i\left[\frac{3}{\sqrt{15}}\varphi_2(\eta)-\frac{2}{\sqrt{35}}\varphi_3(\eta)\right]} \\ H_{100}(\eta) &= e^{i\left[\frac{1}{\sqrt{3}}\varphi_1(\eta)-\frac{2}{\sqrt{15}}\varphi_2(\eta)-\frac{2}{\sqrt{35}}\varphi_3(\eta)\right]}. \end{aligned} \tag{31}$$

Operators for excitations with higher charge are obtained as products of these; for example, the product of all three is a charge-3/7 vortex. Again, it is straightforward to check that the operators (31) span the charge lattice. In direct generalization of the $\nu = 2/5$ case, the $\nu = 3/7$ quasiparticle operator is given by a combination of the inverse hole operator in the highest occupied CF Landau level, *i.e.* H_{001}, and V_3, with one additional derivative,

$$P_{3/7}(z) = \partial^3 e^{i[\frac{2}{\sqrt{3}}\varphi_1(z)+\frac{2}{\sqrt{15}}\varphi_2(z)+\frac{2}{\sqrt{35}}\varphi_3(z)]}. \tag{32}$$

The pattern for construction of higher level operators in the $\nu = s/(2s+1)$ series should now be obvious, and in Appendix B 3 we give the general expressions for the operators V_{pn} describing the electrons at the n^{th} level in the $n/(2np+1)$ series, as well as the corresponding current density operator. The proof that the CF wave functions for n filled CF Landau levels are reproduced by sums of correlators with an equal number of V_{pn}:s (for fixed p) is outlined in Appendix B 3. The construction of the pertinent quasihole operators should be straightforward, although we have not derived the explicit formulae beyond the ones given above.

From the general expressions of the operators, it is easy to see that two operators $V_{pn}(z_i)$ and $V_{pn}(z_j)$ at the same level give a factor $(z_i - z_j)^{2p+1}$ in the correlation function, while two operators $V_{p,n_1}(z_i)$ and $V_{p,n_2}(z_j)$ at different levels produce a factor $(z_i - z_j)^{2p}$ (see appendix B 3). This gives an alternative way to calculate the filling fraction, and also demonstrates that the limiting value for $n \to \infty$ is $\nu = 1/2p$.

IV. CONNECTION TO EFFECTIVE CHERN-SIMONS THEORIES AND EDGE STATES

Wen has developed a general effective theory formalism for the QH liquids based on representing the currents by two dimensional gauge fields $a_{I\mu}$ with a Chern-Simons action[3],

$$\mathcal{L} = -\frac{1}{4\pi} K_{II'} a_{I\mu} \partial_\nu a_{I'\lambda}\, \varepsilon^{\mu\nu\lambda} - \frac{e}{2\pi} A_\mu \partial_\nu t_I a_{I\lambda} \varepsilon^{\mu\nu\lambda}, \tag{33}$$

where the matrix K and the "charge vector" $\mathbf{t}^T = (t_1 \dots t_p)$ have integer elements. The filling fraction is given by $\nu = \mathbf{t}^T K^{-1} \mathbf{t}$. A generic quasiparticle carries integral charges of the $a_{I\mu}$ field, and is thus labeled by p integers constituting the vector $\mathbf{l} = (l_1 \dots l_p)$. The electric charge and the statistics of the quasiparticle are given by $q = -e\mathbf{t}^T K^{-1}\mathbf{l}$ and $\theta = \pi \mathbf{l}^T K^{-1} \mathbf{l}$, respectively. This description is not unique; as explained in reference [3], an equivalent description is given by $(K', \mathbf{t}', \mathbf{l}') = (WKW^T, W\mathbf{t}, W\mathbf{l})$ where W is an element of $SL(p, Z)$, *i.e.* an integer valued $p \times p$ matrix with unit determinant.

As an example of the above, the $\nu = 2/5$ state is described by the K matrix and $\mathbf{t}$ vector,

$$K_{2/5} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \qquad \mathbf{t}^T = (1,1)\,. \tag{34}$$

This is an example of what Wen refers to as the symmetric basis, where in general $\mathbf{t}^T = (1,1,\ldots,1)$. By an $SL(2,Z)$ transformation, we can represent the same state in the "hierarchy basis" (which naturally occurs when constructing states in the Halperin-Haldane hierarchy) characterized by $\mathbf{t}^T = (1,0,\ldots,0)$.

$$K'_{2/5} = WKW^T = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \quad ; \quad \mathbf{t}'^T = \mathbf{t}^T W^T = (1,0) \quad ; \quad W = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \tag{35}$$

Starting from the Chern-Simons theory (33) defined on a finite two dimensional domain, one can derive a dynamical theory for the edge excitations. The details can be found in [3] and references therein, and the resulting theory is

$$S_{ed} = \frac{1}{4\pi}\int dtdx\,[K_{IJ}\partial_t\phi_I\partial_x\phi_J - V_{IJ}\partial_x\phi_I\partial_x\phi_J + 2eA_\mu\epsilon^{\mu\nu}\partial_\nu t_I\phi_I], \tag{36}$$

where K and $\mathbf{t}$, as well as the quasiparticle vector $\mathbf{l}$, are the same as in the effective bulk theory (33). This is a multicomponent chiral boson theory with the current operator given by

$$J^\mu = -\frac{\delta S}{\delta A_\mu} = -\frac{e}{2\pi}\epsilon^{\mu\nu}t_I\partial_\nu\phi_I\,. \tag{37}$$

The quasiparticle operators (including the electron operator) take the generic form

$$\Psi \sim e^{i\sum_q l_q\phi_q}\,, \tag{38}$$

familiar from abelian bosonization of one-dimensional fermion systems. The numbers V_{IJ} are the non-univeral edge velocities, which depend on the details of the confining potential.

In their original paper on the connection between QH liquids and conformal field theories, Moore and Read made two basic claims. The first, which we already have discussed, is that the electronic wave functions can be expressed as conformal blocks of certain CFT:s. The second is that this very same CFT *is* the one dimensional theory describing the dynamical edge excitations. This last claim should not be taken literally since it is known that the edge dynamics is non-universal. Not only the edge velocities, but also the character, and even the number of edge modes can depend on details of the edge potential. Examples are the polarization edge modes related to edge spin texture[27] and the counter-propagating modes resulting from edge reconstruction as first discussed by Shamon and Wen[28]. Thus we can only hope that the CFT will provide a "minimal" edge theory consistent with the topological properties of the bulk, *i.e.* that it supports excitations with the same charges. In spite of these limitations, the Moore-Read conjecture about the edge theory has been very fruitful, especially in the search for effective field theories for the non-abelian Pfaffian state[29].

We shall now demonstrate the connection between the CFT construction of the Jain states and Wen's K-matrix formulation by explicitly working out the case of $\nu = 2/5$. Led by the Moore-Read conjecture, we will start from our CFT bulk theory, read off the K-matrix and the charge vector, and show that in the basis where (26) are the fundamental quasihole operators, one exactly recovers Wen's K-matrix and $\mathbf{t}$-vector in the symmetric basis. This is consistent with Read's earlier result[30] that the symmetric basis naturally describes the Jain states. Alternatively, we may choose a basis consisting of either of the charge 1/5 quasiholes in (26), along with the charge 2/5 vortex (*i.e.* the product of the two 1/5-hole operators); as we shall see, this instead corresponds to the hierarchical basis.

The conformal field theory contains the two uncoupled bosonic fields φ_1 and φ_2, compacitfied on radii $R^2 = 3$ and 15, respectively. The corresponding action, $S = \int d^2x\,\mathcal{L}_{cft}$ for the full scalar fields $\phi_i(x,t) = \varphi_i(z) + \bar{\varphi}_i(\bar{z})$ is obtained from the Lagrangian,

$$\mathcal{L}_{cft} = \frac{1}{8\pi}(\bar{\phi}_1,\bar{\phi}_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\partial_\mu\partial^\mu\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{e}{2\pi}\tilde{t}_I A^\mu\,\epsilon_{\mu\nu}\partial_\nu\phi_I \equiv \frac{1}{4\pi}K_{IJ}\phi_I\partial_\mu\partial^\mu\phi_J - A^\mu J_\mu\,. \tag{39}$$

where the information about the compactification radii is contained in the charge vector $\tilde{\mathbf{t}}^T = (1/\sqrt{3}, 1/\sqrt{15})$. The Lagrangian (39) contains both right and left moving fields, but these decouple, and it is known that the dynamics of a single chiral component, such as $\varphi_i(z)$, is described by the first order Lagrangian (36) with the same K matrix and $\mathbf{t}$ vector[31]. In order to directly compare with Wen's formalism, we rescale the Bose fields such as to obtain an integer charge vector, $\mathbf{t}^T = (1,1)$: $(\varphi'_1,\varphi'_2) \equiv (\varphi_1/\sqrt{3}, \varphi_2/\sqrt{15})$. Naively, the corresponding K matrix would then be

$diag(3,15)$. It is however important to remember that a CFT is not defined only by the Lagrangian of the fields φ_i, which gives the operator product expansions, or fusion rules, of the primary fields (*i.e.* the vertex operators), but also by primary field content, *i.e.* the allowed vertex operators. In the case of the $\nu = 2/5$ state these allowed fields define a charge lattice with the basis vectors given by the quasihole operators (26). Thus, we will change to a basis (χ_1, χ_2) where the fundamental quasihole operators spanning the charge lattice are given by $H_i = e^{i\chi_i}$. As can be seen from (26), this is achieved by the field redefinition,

$$\begin{aligned} \chi_1 &= \frac{3}{\sqrt{15}}\,\varphi_2 = 3\varphi_2' \\ \chi_2 &= \frac{1}{\sqrt{3}}\,\varphi_1 - \frac{2}{\sqrt{15}}\,\varphi_2 = \varphi_1' - 2\varphi_2'\,. \end{aligned} \tag{40}$$

Inverting this transformation and inserting into (39), it is now easy to verify that the resulting K matrix and $\mathbf{t}$ vector are precisely the K and $\mathbf{t}$ in (34). Alternatively (and equivalently), if we start from a basis of one of the 1/5 quasihole operators, say H_{10}, together with the charge 2/5 "vortex" $H_{11} \equiv H_{10}H_{01}$, corresponding to the change of basis

$$\begin{aligned} \chi_1 &= \varphi_1' + \varphi_2' \\ \chi_2 &= \varphi_1' - 2\varphi_2'\,, \end{aligned} \tag{41}$$

we find that the corresponding K-matrix and $\mathbf{t}$-vector are the ones given in (35), *i.e.* the hierarchical basis. This equivalence, at the effective Chern-Simons theory level, of the Jain states and the hierarchy scheme, has been previously pointed out by several authors[30,32]. These authors arrive at this result by a general argument, based on similarity between the Jain states and filled Landau levels, that ignores the projection on the lowest Landau level. It is reassuring that the above demonstration, based on explicitly holomorphic wave functions, leads to the same result.

This construction straightforwardly carries over to the other fractions in the Jain sequence; for example, in the case of $\nu = 3/7$, one may pick the three charge-1/7 quasihole operators of (31) as basis of the charge lattice, corresponding to the field redefinition

$$\begin{aligned} \chi_1 &= \frac{5}{\sqrt{35}}\varphi_3 \\ \chi_2 &= \frac{3}{\sqrt{15}}\varphi_2 - \frac{2}{\sqrt{35}}\varphi_3 \\ \chi_3 &= \frac{1}{\sqrt{3}}\varphi_1 - \frac{2}{\sqrt{15}}\varphi_2 - \frac{2}{\sqrt{35}}\varphi_3\,. \end{aligned} \tag{42}$$

Again, this brings us to the symmetric basis, with $\mathbf{t}^T = (1,1,1)$ and the K-matrix given by $K_{ij} = 2+\delta_{ij}$. Alternatively, we may construct a basis consisting of quasihole operators with charge 1/7, 2/7 and 3/7, respectively, by appropriate combinations of the charge-1/7 quasiholes in (31). As before, this corresponds to the hierarchical basis, with $\mathbf{t}^T = (1,0,0)$ and the same K-matrix as that given by Wen[3],

$$K^h_{3/7} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{43}$$

V. LOCALIZED QUASIPARTICLES AND FRACTIONAL CHARGE AND STATISTICS

The present formulation already gives a strong hint for fractional charge and fractional statistics of the CF quasiparticles: We have seen from (7) that the operator $P_{\frac{1}{m}}(z)$ corresponds to a localized charge at z, and the presence of the factor $(z_i - z_j)^{\frac{1}{m}}$ is suggestive of fractional statistics with angle $\frac{\pi}{m}$. This is not a proof, however. The usual argument for fractional charge and statistics proceeds via the Berry phases produced by adiabatic braidings of localized quasiparticles. In this section we construct the wave functions for localized states of one and two quasiparticles, and use these to calculate the Berry phases relevant for charge and statistics within what we call a "random phase assumption".

A localized quasiparticle state is constructed as a coherent superposition of a the angular momentum states given

in (11) and (15). For a single quasiparticle at location $\bar{\eta}$ we have (putting $\ell = 1$)

$$\begin{aligned}\Psi_{1qp}(\eta,\bar{\eta};z_i) &= \tilde{\mathcal{N}_1}(\bar{\eta}\eta)e^{-\frac{1}{4m}|\bar{\eta}|^2}\sum_{l=0}^{\infty}\frac{\bar{\eta}^l}{(2m)^l l!}\Psi^l_{1qp}(z_i) \\ &= \tilde{\mathcal{N}_1}(\bar{\eta}\eta)\sum_i(-1)^i e^{-\frac{1}{4m}(|z_i|^2+|\bar{\eta}|^2-2\bar{\eta}z_i)}\langle P(z_i)\prod_j{}^{(i)}V(z_j)\rangle\,.\end{aligned} \tag{44}$$

Notice that the normalization constant $\mathcal{N}_1(\bar{\eta}\eta) = \tilde{\mathcal{N}_1}(\bar{\eta}\eta)e^{-\frac{1}{4m}|\bar{\eta}|^2}$ only depends on the combination $\bar{\eta}\eta$. Likewise, we construct the wave function for two quasiparticles at positions $\bar{\eta}_\pm = \bar{N} \pm \bar{\eta}/2$ as

$$\Psi_{2qp}(N,\bar{N},\eta,\bar{\eta};z_i) = \tilde{\mathcal{N}_2}(N,\bar{N},\eta,\bar{\eta})e^{-\frac{1}{8m}|\bar{\eta}|^2-\frac{1}{2m}|\bar{N}|^2}\sum_{l=1,3,\ldots}\sum_{L=0,1,\ldots}\frac{(\frac{1}{m}\bar{N})^L}{L!}\frac{(\frac{1}{4m}\bar{\eta})^{l-1}}{l!}\Psi^{l,L}_{2qp}(z_i) \tag{45}$$

$$= \tilde{\mathcal{N}_2}\frac{4m}{\bar{\eta}}\sum_{i<j}(-1)^{i+j}e^{-\frac{1}{8m}(|\bar{\eta}|^2+|z_{ij}|^2)}\sinh\frac{\bar{\eta}z_{ij}}{4m}e^{-\frac{1}{2m}(|\bar{N}|^2+|Z_{ij}|^2-2\bar{N}Z)}z_{ij}^{1-\frac{1}{m}}\,\langle P_{\frac{1}{m}}(z_i)P_{\frac{1}{m}}(z_j)\prod_k{}^{(ij)}V_1(z_k)\rangle\,,$$

where $z_{ij} = z_i - z_j$. For $\bar{\eta} = 0$ and $\bar{N} = \bar{\eta} = 0$, respectively, these expressions reduce to Ψ^0_{1qp} and $\Psi^{0,0}_{2qp}$, the wave functions with minimum angular momentum. The explicit wave functions obtained by evaluating the correlators in (44) and (45), are very similar, but not identical, to the corresponding CF wave functions. One source of difference is the slight deviation between the angular momentum eigenstates given by (15) and (16), pointed out in section II C, and shown to be numerically insignificant in section VI. The other source of difference can be seen already for the one quasiparticle state. The CF wave function reads

$$\Psi_{CF} = \sum_i(-1)^i\, e^{-\frac{|\eta|^2}{4m}+\frac{\bar{\eta}z_i}{2m}}\prod_{j<k}^{(i)}(z_j-z_k)^3\prod_n^{(i)}(z_i-z_n)^2\left[\left(\frac{1}{m}-1\right)\bar{\eta}+\sum_n^{(i)}\frac{4}{z_i-z_n}\right]\exp\left(-\frac{1}{4}\sum_k|z_k|^2\right), \tag{46}$$

but the term proportional to $(1/m-1)\bar{\eta}$ is missing in the corresponding CFT wave function. This difference, however, is a finite size effect. This term contributes to the wave function only when the exponential factor $\exp(\bar{\eta}z_i/2m)$ is expanded to the N^{th} power, and thus amounts to a (nonuniversal) boundary term.

Before proceeding to the calculation of the charge and statistics of the quasiparticles, it is helpful to recall, as a background, the corresponding calculation for the quasiholes of $\nu = 1/m$. Consider the normalized wave function for one quasihole, given by (9):

$$\Psi(\eta,\bar{\eta};z_i) \;=\; \mathcal{N}'e^{-\frac{1}{4m\ell^2}|\eta|^2}\Psi_L(\eta;z_i) \tag{47}$$

where $\Psi_L(\eta;z_i) = \prod_i(z_i-\eta_1)\prod_{k<l}(z_k-z_l)^m e^{-\frac{1}{4\ell^2}\sum_i|z_i|^2}$ is the Laughlin unnormalized wave function for a single quasihole. The plasma analogy shows that $\mathcal{N}'$ is independent of η; the normalization integral of (47) is the partition function of a Coloumb plasma with a charged impurity, which is independent of the position of the impurity as long as it is farther than a screening length (ℓ) from the edge. The Berry phase associated with a circular loop $\eta = Re^{i\theta};\ \theta \in \{0, 2\pi\}$, is given by (with $\hbar = c = 1$)

$$\begin{aligned}\gamma_B &= \int_0^{2\pi}d\theta\,\langle\Psi(\eta,\bar{\eta})|i\partial_\theta|\Psi(\eta,\bar{\eta})\rangle \\ &= \int_0^{2\pi}d\theta\,\langle\Psi(\eta,\bar{\eta})|(\bar{\eta}\partial_{\bar{\eta}}-\eta\partial_\eta)|\Psi(\bar{\eta},\eta)\rangle \\ &= -A\frac{e}{m}B,\end{aligned} \tag{48}$$

where A is the area enclosed by the loop, and the last line employs integration by parts and the observation that the only $\bar{\eta}$ dependence in the wave function is from the gaussian factor. The result, as expected, is the Aharanov-Bohm phase for a particle of charge e/m.

Turning to two quasiholes and again using (9) we have

$$\Psi(\bar{\eta}_a,\eta_a;z_i) = \mathcal{N}''(\eta_1-\eta_2)^{\frac{1}{m}}e^{-\frac{1}{4m\ell^2}(|\eta_1|^2+|\eta_2|^2)}\Psi_L(\eta_a;z_i) \tag{49}$$

where $\Psi_L(\eta_a;z_i) = \prod_i(z_i-\eta_1)\prod_j(z_j-\eta_2)\prod_{k<l}(z_k-z_l)^m e^{-\frac{1}{4\ell^2}\sum_i|z_i|^2}$ is the unnormalized Laughlin wave function for two quasiholes. Naively we would read the fractional statistics parameter as $\theta = \pi/m$ directly form the factor

$(\eta_1-\eta_2)^{\frac{1}{m}}$ in (49) but that gives the correct result only if there is no extra Berry phase (other than the usual Aharonov-Bohm phase) associated with the exchange path (here integrating θ from 0 to π)[33]. The absence of additional phases can be confirmed by an explicit calculation using the plasma analogy for a system with two impurities separated farther than the screening distance, ℓ_0. Taking $\eta_1 = -\eta_2 = \eta$, we get $\gamma_B^{ex} = -\frac{e}{m}B\pi R^2$, which is nothing but the AB-phase expected from the exchange of two charge $\frac{e}{m}$ particles through a circular path with radius R. That allows us to read the exchange statistics phase directly from the factor $(\eta_1 - \eta_2)^{\frac{1}{m}}$ in (49). Wilczek and Nayak[33] suggest that it is no coincidence that (9) yields a wave function with no Berry contribution to the statistics angle; they make a conjecture, supported by arguments, that QH wave functions given directly as correlators, or conformal blocks, of a CFT have vanishing Berry phase (forgetting the Aharanov-Bohm contribution) so the exchange statistics can be obtained from the so-called monodromy, which in this simple case is just the phase $e^{i2\pi/m}$ produced by the factor $(\eta_1 - \eta_2)^{\frac{1}{m}}$ when the quasiholes braid around each other along closed paths defined via an analytic continuation of the original wave function. In the more general case of non-Abelian fractional statistics, several conformal blocks correspond to the same configuration of quasihole coordinates, and the monodromies are now matrices that encode how these conformal blocks transform into each other under braidings of the coordinates.

Rather than using (48), we follow Kjønsberg and Leinaas who showed that for a (normalized) wave function of the form, $\Psi(\eta,\bar\eta;z_i) = \mathcal{N}(\bar\eta\eta)\Psi_h(\eta,\bar\eta;z_i)$, where $\mathcal{N}$ is a real function only of $\bar\eta\eta$, the Berry phase is given by[34]

$$\gamma_B = \int_0^\Theta d\theta\, \langle\Psi(\eta,\bar\eta)|i\partial_\theta|\Psi(\eta)\rangle = \Theta R^2 \frac{d}{dR^2}\ln\mathcal{N}(\bar\eta\eta)^2\,. \tag{50}$$

The Laughlin wave function for a single quasihole at η or two quasiholes at $\pm\eta$ has this form. The upper limit is taken to be $\Theta = 2\pi$ for a single quasihole, and $\Theta = \pi$ for an exchange of two quasiholes. From (47) and (49), the normalization constants are given by (up to an η independent factor)

$$\begin{aligned}\mathcal{N}_1(\bar\eta\eta) &= e^{-\frac{1}{4m\ell^2}|\eta|^2} \\ \mathcal{N}_2(\bar\eta_i\eta_i) &= |\eta_1-\eta_2|^{\frac{1}{m}} e^{-\frac{1}{4m\ell^2}(|\eta_1|^2+|\eta_2|^2)},\end{aligned} \tag{51}$$

so the formula (50) can be applied. (We have assumed sufficiently far separated quasiholes.) For a single quasihole loop of radius R, (50) gives the Berry phase $-2\pi AB(e/m)$. For two quasiholes at $\pm\eta$, the Berry phase is $(\pi/m) - AB(e/m)$ (with $\Theta = \pi$). The difference, π/m, gives the contribution from fractional statistics. In this case, there is no monodromy, and the the full statistical phase appears as a Berry phase.[48]

We now turn to the case of quasiparticles, where we need to calculate the relevant normalization constants from the wave functions (44) and (45). Because the sum in (45) extends only over even powers of $\bar\eta$, the holomorphic part is single valued under $\bar\eta \to -\bar\eta$, implying an absence of monodromy contribution to the statistical angle, and thus both the charge and statistics can be extracted directly from $\mathcal{N}_1$ and $\mathcal{N}_2$, provided that they can be chosen as real functions of $\bar\eta\eta$ only. For quasiparticles, the normalized wave function has the form $\mathcal{N}(\bar\eta\eta)\Psi(\bar\eta)$, and the Berry phase is given by

$$\gamma_B = -\Theta R^2 \frac{d}{dR^2}\ln\mathcal{N}(\bar\eta\eta)^2\,. \tag{52}$$

Unfortunately, the calculation of the normalization factors is more difficult than in the case of the quasiholes, because the electronic wave functions (44) and (45) involve sums over correlators; *e.g.* for two quasiparticles the relevant integral is $\sim \sum_{ijkl}(-1)^{i+j+k+l}\langle P_{\frac{1}{m}}(z_i)P_{\frac{1}{m}}(z_j)\prod_p{}^{(ij)}V_1(z_p)\rangle\langle P_{\frac{1}{m}}(\bar z_k)P_{\frac{1}{m}}(\bar z_l)\prod_q{}^{(lm)}V_1(\bar z_j)\rangle$. If, however, we keep only the diagonal terms in the sums, which amounts to a kind of random phase assumption discussed in the next section, then the normalization factors can be calculated as shown in Appendix C. The calculation of the normalization constants, outlined in Appendix C, gives the result:

$$\mathcal{N}_1^2 = \tilde{\mathcal{N}}_1^2\, e^{-\frac{1}{2m\ell^2}|\eta|^2} \sim (R^2)^{-1}e^{-\frac{1}{2m\ell^2}R^2} \quad ; \quad \eta = Re^{i\theta} \tag{53}$$

$$\mathcal{N}_2^2 = \tilde{\mathcal{N}}_2^2\, e^{-\frac{1}{2m\ell^2}|\eta|^2} \sim (R^2)^{\frac{1}{m}-2}e^{-\frac{1}{4m\ell^2}R^2} \quad ; \quad \eta = \frac{R}{2}e^{i\theta} \tag{54}$$

Using (52) we can extract the fractional charge from the Berry phase corresponding to a single quasiparticle,

$$\gamma_B = \frac{e}{m}B\pi(R^2 + 2m\ell^2) = \frac{e}{m}BA + 2\pi\,, \tag{55}$$

so the leading term is precisely the expected AB phase for a charge e/m object. The $O(\ell^2)$ term is a quantum correction to the "classical" R^2. Such corrections have been discovered earlier both in the context of CF quasiparticles[16] and

noncommutative matrix models[35]. The statistical angle is obtained by subtracting this result from the Berry phase extracted from the exchange path. Remembering that the effective area enclosed by the exchange path is $\pi R^2/4$, we get

$$\theta = \gamma_B^{exc} - \frac{1}{4}\gamma_B = -\frac{\pi}{m}\,. \tag{56}$$

This reproduces the result obtained from a direct numerical evaluation of the Berry phases from the composite fermion wave functions (44) and (45)[15,16,19]. The statistics of the quasiholes and quasiparticles obtained above differ in sign, contrary to the expectation from general considerations based on effective Chern-Simons or the CF theory, and we want to comment upon this. Both normalization constants in (53) are of the form $\mathcal{N} \sim (R^2)^a e^{-bR^2}$, and from the derivation in the Appendix one sees that the constant a is unambiguously determined, and the same holds for the fractional part of b, while the integer part of b could in principle be shifted by using a different prescription for the ordering of the derivatives. However, this does not necessarily mean that the frational part of θ is well determined since it is sensitive to a cancellation of a large number of terms proportional to R^2 in the Berry phases. The above result is based on the assumption that we have correctly identified $|\eta|$ as the distance between the two quasiparticles. This question is discussed in some detail in in Ref. 16 where it is shown that the distance between two quasiparticles is slightly different from $|\eta|$; correcting for the distance produces the same statistics as for quasiholes (modulo an integer). Because our wave function is closely related to the CF wave function, similar considerations should apply here as well.

VI. NUMERICAL TESTS

This section concerns quantitative tests of the CFT quasiparticle wave functions. The numerical tests consist of two parts. In the first part, we compare, at filling factor $\nu = 1/3$, the two-quasiparticle wave functions (15) obtained from the conformal field theory with the standard wave functions from the CF theory by calculating their Coulomb interaction energies and overlap. We will find that the two are practically identical. In the second part, the random phase approximation used in section V is examined.

A. Two-quasiparticle wave function

The N-composite fermion wave function for two quasiparticles at filling factor $\nu = 1/3$ is constructed by compactly filling the lowest CF-LL by $N-2$ composite fermions, and placing the remaining two composite fermions in the second CF-LL. We consider below the state in which the two "excited" composite fermions are in angular momentum orbitals, and occupy the smallest angular momenta. The wave function for this state is written as

$$\Psi_{\rm CF} = \mathcal{P}_{\rm LLL} \begin{vmatrix} \eta_{1,-1}(z_1) & \eta_{1,-1}(z_2) & \dots & \eta_{1,-1}(z_N) \\ \eta_{1,0}(z_1) & \eta_{1,0}(z_2) & \dots & \eta_{1,0}(z_N) \\ \eta_{0,0}(z_1) & \eta_{0,0}(z_2) & \dots & \eta_{0,0}(z_N) \\ \eta_{0,1}(z_1) & \eta_{0,1}(z_2) & \dots & \eta_{0,1}(z_N) \\ \vdots & \vdots & & \vdots \\ \eta_{0,N-3}(z_1) & \eta_{0,N-3}(z_2) & \dots & \eta_{0,N-3}(z_N) \end{vmatrix} \times \prod_{i<j}(z_i - z_j)^2 \exp\left(-\frac{1}{4}\sum_k |z_k|^2\right), \tag{57}$$

where $\mathcal{P}_{\rm LLL}$ denotes projection into the lowest Laudau level (LLL), and $\eta_{n,m}(z)$ is the single particle eigenstate in symmetric gauge:

$$\eta_{n,m}(z,\bar{z}) = N_{n,m}\, e^{-|z|^2/4} \sum_{k=k_0}^{n} (-1)^k \begin{pmatrix} n+m \\ n-k \end{pmatrix} \frac{1}{2^k k!} \bar{z}^k z^{k+m}, \tag{58}$$

in which $n = 0, 1, 2, \dots$ labels the Landau level index, $m = -n, -n+1, -n+2, \dots$ is the angular momentum quantum number, $k_0 = \max(0, -m)$, and

$$N_{n,m} = \sqrt{\frac{n!}{2\pi 2^m (n+m)!}} \tag{59}$$

is the normalization constant. As usual, the magnetic length has been set to unity. The wave function can be shown to be equal to

$$\Psi_{\rm CF} = \mathcal{P}_{\rm LLL} \begin{vmatrix} \bar{z}_1 z_1 & \bar{z}_2 z_2 & \dots & \bar{z}_N z_N \\ \bar{z}_1 & \bar{z}_2 & \dots & \bar{z}_N \\ 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ \vdots & \vdots & & \vdots \\ z_1^{N_3} & z_2^{N_3} & \dots & z_N^{N_3} \end{vmatrix} \times \prod_{i<j} (z_i - z_j)^2 \exp\left(-\frac{1}{4} \sum_k |z_k|^2 \right). \tag{60}$$

Following the standard procedure, the projection procedure is accomplished by expanding the determinant, moving all $\bar{z}$'s to the left, and replacing $\bar{z}$ by $2\partial/\partial z$ (with the convention that the derivatives do not act on the Gaussian part). A technique developed in Ref. [36] has made it possible to perform the projection in a more convenient manner, which is the one we use below. (This method gives projected states very close to those obtained by "brute force" projection.) The projected wave function is written as

$$\begin{aligned} \Psi_{\rm CF} &= \sum_{i<j} (-1)^{i+j} (z_i - z_j)^3 \prod_k^{(ij)} (z_i - z_k)^2 \prod_l^{(ij)} (z_j - z_l)^2 \prod_{m<n}^{(ij)} (z_m - z_n)^3 \\ &\times \left\{ \frac{-2}{(z_i - z_j)^2} + \sum_{k,l}^{(ij)} \frac{4}{(z_i - z_k)(z_j - z_l)} + \sum_k^{(ij)} \frac{4}{(z_k - z_i)(z_k - z_j)} \right\} \exp\left(-\frac{1}{4} \sum_k |z_k|^2 \right). \end{aligned} \tag{61}$$

The symbol (ij) denotes that indices i and j are excluded in the summation or the product. The total angular momentum of (61) is

$$L = \frac{3}{2} N^2 - \frac{7}{2} N + 2. \tag{62}$$

Numerical simulations for the wave function in (61) have shown that it produces better variational energy than the two quasiparticles wave function obtained by generalizing Laughlin's single quasiparticle state[25].

The CFT wave function for two quasiparticles located at the origin is given by (14). To make contact with the above CF wave function, we set the center of mass angular momentum to zero and put the two quasiparticles in the smallest relative angular momentum channel $l = 1$; that produces a wave function that has the total angular momentum given in (62) . For these parameters, the CFT ansatz for the two quasiparticles wave function $\Psi_{\rm CFT}$ at $\nu = 1/3$ reduces to

$$\begin{aligned} \Psi_{\rm CFT} &= \sum_{i<j} (-1)^{i+j} (z_i - z_j)^3 \prod_k^{(ij)} (z_k - z_i)^2 \prod_l^{(ij)} (z_l - z_j)^2 \prod_{m<n}^{(ij)} (z_m - z_n)^3 \\ &\times \left\{ -\frac{4}{9} \frac{1}{(z_i - z_j)^2} + \frac{8}{3} \sum_k^{(ij)} \frac{1}{(z_k - z_i)(z_k - z_j)} + \sum_{k,l}^{(i,j)} \frac{4}{(z_k - z_i)(z_l - z_j)} \right\} \exp\left(-\frac{1}{4} \sum_k |z_k|^2 \right). \end{aligned} \tag{63}$$

To compare the two-quasiparticle wave functions in Eqs. (63) and (61), we compare their Coulomb interaction energies and also calculate their overlaps. The Coulomb energy (in units of $e^2/\epsilon\ell_0$) is defined as

$$E = \frac{\left\langle \Psi \left| \frac{1}{2} \sum_{i \neq j} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} \right| \Psi \right\rangle}{\langle \Psi | \Psi \rangle}, \tag{64}$$

We do not include here electron-background and background-background contributions; these are not necessary for the present purpose, as they are identical for the two wave functions. The overlap is defined as

$$\mathcal{O} = \frac{|\langle \Psi_{\rm CFT} | \Psi_{\rm CF} \rangle|}{\sqrt{\langle \Psi_{\rm CFT} | \Psi_{\rm CFT} \rangle \langle \Psi_{\rm CF} | \Psi_{\rm CF} \rangle}}. \tag{65}$$

Both quantities are evaluated using Metropolis Monte Carlo integration. A single data point is obtained by averaging 100 independent Monte Carlo runs, with $\sim 1.2 \times 10^5$ iterations in each run. For $N = 50$ electrons, the total

N	$E_{\rm CFT}$	$E_{\rm CF}$	$\mathcal{O}$
10	7.76619(62)	7.76600(62)	0.9999301(2)
20	24.1403(19)	24.1402(19)	0.9999274(4)
30	46.3258(18)	46.3257(18)	0.9999274(3)
40	73.2339(17)	73.2339(17)	0.9999266(2)
50	102.0588(16)	102.0585(16)	0.999613(5)

TABLE I: The Coulomb energy $E_{\rm CFT}$ ($E_{\rm CF}$), quoted in units of $e^2/\epsilon\ell$, for the CFT (composite fermion) two-quasiparticle wave function $\Psi_{\rm CFT}$ ($\Psi_{\rm CF}$). $\mathcal{O}$ is the properly normalized overlap between the two candidate wave functions for a number of system sizes N. In $\Psi_{\rm CFT}$, we set $L = 0$ and $l = 1$. The definitions of energy and overlap are given in the text. The Monte Carlo statistical uncertainty is shown in brackets.

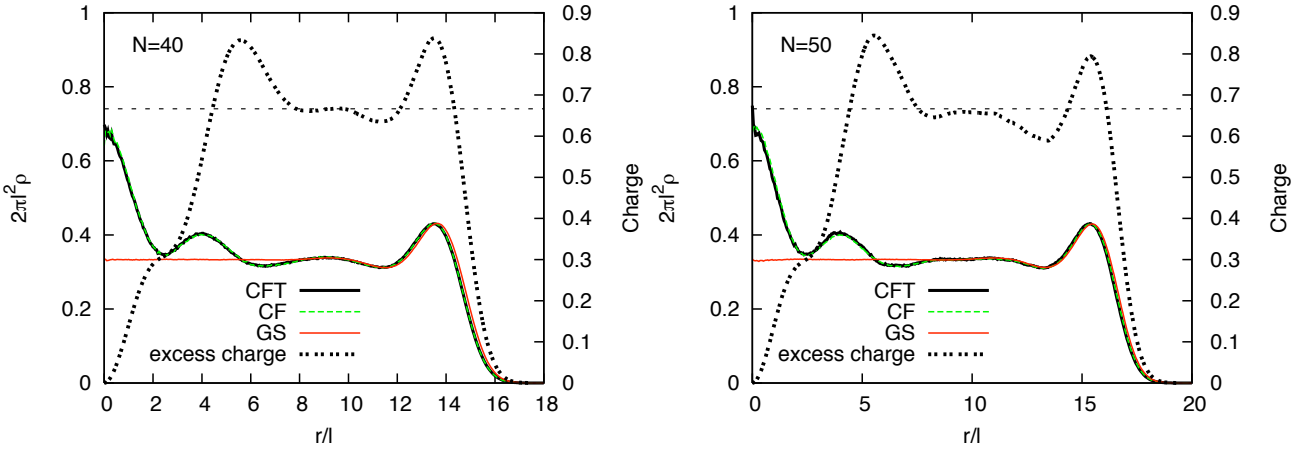

FIG. 1: Density profiles of $\Psi_{\rm CFT}$ and $\Psi_{\rm CF}$ for $N = 40$ (left panel), and 50 (right panel) particles. "GS" denotes the $\nu = 1/3$ background obtained from Laughlin's wave function. The horizontal dashed line indicates 2/3 units of charge. In both cases, the excess integrated charge for $\Psi_{\rm CFT}$, denoted by thick dashed line, shows the correct quantized value. The statistical undertainty in Monte Carlo is smaller than the widths of the lines.

computational time is approximately 100 hours on a single node of a Beowulf-type PC cluster consisting of dual 3.06 GHz Intel Xeon Processors. The results are summarized in Table I. The excellent agreement demonstrates that the two two-quasiparticle wave functions are essentially identical. Figure 1 depicts the density profiles of the two wave functions for $N = 40$ and 50 electrons. The excess integrated charge (*i.e.* the charge measured relative to the $\nu = 1/3$ background) for the CFT wave function is also shown. The excess charge has a well-defined plateau at 2/3 before the edge distortion. (The creation of two quasiparticles near the center of the droplet induces two quasiholes of equal charge at the edge.)

B. Random Phase Approximation

The composite fermion quasiparticles have been shown to possess well-defined fractional braiding statistics[16]. The comparisons in the previous subsection imply that this property, in principle, carries over to the CFT quasiparticles. However, one of the strengths of the CFT description of the FQHE states is that it reveals the braiding properties in a transparent manner.

A calculation of the braiding statistics requires wave functions for spatially localized quasiparticles, which are constructed in the previous section. A key observation is that the braiding statistics can be obtained from the wave function normalization factor, which depends only on the coordinates of localized quasiparticles (c.f. (45)). However it is difficult to obtain an explicit analytical expression for the normalization factor for localized quasiparticle states, because the localized quasiparticle wave functions are sums over many correlators; for example, the relevant term in the integral for the normalization factor of a single quasiparticle is

$$\sum_{ij}\langle P_m(z_i)\prod_{k\neq i}V_m(z_k)\rangle\langle P_m(\bar{z}_j)\prod_{l\neq j}V_m(\bar{z}_l)\rangle. \tag{66}$$

The analytic form of the normalization factor can be obtained by assuming that only the "diagonal" elements in (66) are relevant, which we have referred to as the random-phase assumption; the braiding properties of the quasiparticles can be derived under this approximation and are in agreement with the known results. In this section we test the random phase assumption for a single localized quasiparticle.

The wave function for a single quasiparticle at $\nu = 1/m$ localized at $\bar{\eta}$ is written as

$$\Psi_{1\mathrm{qp}}(\bar{\eta}) = \sum_i (-1)^i \, e^{-\frac{1}{4m}(|\bar{\eta}|^2 + 2\bar{\eta} z_i)} \langle P_m(z_i) \prod_{j \neq i} V_m(z_j) \rangle, \tag{67}$$

with the correlator $\langle P_m(z_i) \prod_{j \neq i} V_m(z_j) \rangle$ given by

$$\langle P_m(z_i) \prod_{j \neq i} V_m(z_j) \rangle = \prod_{j<k}^{(i)} (z_j - z_k)^m \prod_{j \neq i} (z_i - z_j)^{m-1} \sum_{j \neq i} \frac{m-1}{z_i - z_j} \cdot \exp\left(-\frac{1}{4} \sum_l |z_l|^2 \right). \tag{68}$$

For simplicity, we define the notation

$$F(z_i) \equiv (-1)^i \, e^{-\frac{1}{2m}(\bar{\eta} z_i)} \langle P_m(z_i) \prod_{j \neq i} V_m(z_j) \rangle. \tag{69}$$

Then the wave function $\Psi_{1\mathrm{qp}}(\bar{\eta})$ can be expressed as

$$\Psi_{1\mathrm{qp}}(\bar{\eta}) = e^{-\frac{1}{4m}|\bar{\eta}|^2} \sum_i F(z_i). \tag{70}$$

The square of the normalization factor, $\mathcal{N}_1(\eta, \bar{\eta})$, is given by the integral

$$\begin{aligned} \mathcal{N}_1^2(\eta, \bar{\eta}) &= \int \prod_k dz_k \, \Psi^*_{1\mathrm{qp}}(\bar{\eta}) \Psi_{1\mathrm{qp}}(\bar{\eta}) \\ &= e^{-\frac{1}{2m}|\bar{\eta}|^2} \int \prod_k dz_k \left\{ \sum_{i=1}^N |F(z_i)|^2 + \sum_{i=1}^N \sum_{i<j} [F^*(z_i)F(z_j) + F(z_i)F^*(z_j)] \right\}. \\ &\equiv \mathcal{M}_{\mathrm{diag}} + \mathcal{M}_{\mathrm{off-diag}}. \end{aligned} \tag{71}$$

$\mathcal{M}_{\mathrm{diag}}$ and $\mathcal{M}_{\mathrm{off-diag}}$ are the "diagonal" and "off-diagonal" contributions to the full normalization factor, respectively. We calculate the ratio $\mathcal{M}_{\mathrm{diag}}/\mathcal{N}_1(\eta, \bar{\eta})^2$ for several quasiparticle locations for $\nu = 1/3$. The result is shown in Fig. 2. The principal conclusion is that the contribution of the single diagonal term is of the same order as that of a large number of off-diagonal terms. Although not conclusive, this suggests that the diagonal term used to calculate the quasiparticle charge and statistics in section V will be dominant, thus providing a partial justification for the neglect of the off-diagonal terms.

VII. SUMMARY AND OUTLOOK

In this paper we have extended the class of QH wave functions that can be expressed as correlators in a conformal field theory to include the quasiparticle states at the Laughlin fractions, as well as all the ground states in the positive Jain sequence and their quasiparticle and quasihole excitations. The connection between the CFT operators and composite fermions was explicitly demonstrated by constructing n fermionic vertex operators $V_{p,n}$, built from n compactified scalar fields, corresponding to the n filled CF Landau levels at $\nu = n/(2np+1)$. For these states we also constructed the fractionally charged excitations: At $\nu = n/(2np+1)$, there are n independent hole operators (corresponding to removing a composite fermion from any one of the n filled CF Landau levels) and one unique quasiparticle operator (corresponding to putting a CF in the empty $(n+1)^{st}$ Landau level). The fermionic vertex operator $V_{p,n}$ at level n is closely related to the quasiparticle operator of the Jain state with $n-1$ filled CF LLs; in this sense, the ground state at the fraction $n/(2pn+1)$ of the Jain sequence may be viewed as a condensate of quasiparticles of the state at $(n-1)/[2p(n-1)+1]$. It should be noted, however, that these quasiparticles obey well defined fractional statistics only in the dilute limit. We also showed that the conformal field theories used to obtain the CF wave functions give precisely the chiral edge theories that are expected from general considerations within the effective field theory scheme developed by Wen, thus giving microscopic support to that approach.

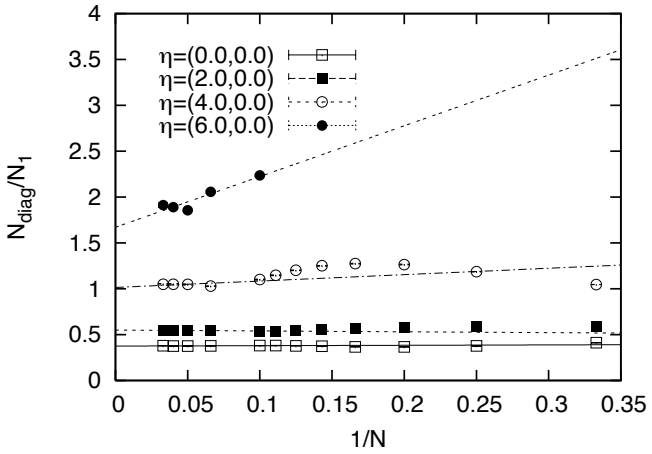

FIG. 2: The ratio $\mathcal{M}_{\rm diag}/\mathcal{N}_1(\eta,\bar\eta)^2$ for several quasiparticle location η. The coordinates are in units of magnetic length. The results for $\eta=(6.0,0.0)$ are not plotted for $N<10$ because of significant edge effects.

An attractive aspect of the methods developed in this paper is that they can be extended and applied to other quantum Hall states. For example, in a recent paper, a straightforward generalization of our vertex operators was employed[38] to describe the states observed recently by Pan *et.al.*[39], which do not belong to the principal series $\nu=n/(2pn+1)$ but have been modeled as the FQHE of composite fermions[39,40]. Generalizing our methods to include the negative Jain sequence, $\nu=n/(2np-1)$, or more generally to states obtained from condensing holes, is a more challenging problem. Another interesting challenge is to find a CFT operator that directly creates a localized quasiparticle, rather than having to construct it as a coherent superposition of angular momentum states as was done in this paper; work on this question is in progress[24].

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APPENDIX A: THE BACKGROUND CHARGE

Definition of correlators of vertex operators such as (8) requires introduction of a compensating charge to satisfy the neutrality condition implied by the conserved $U(1)$ charge[37]. This can be done in different ways. The simplest is to put a compensating charge $V_{bg}(z_\infty)=e^{-i\sqrt{Nm}\varphi(z_\infty)}$ at the position z_∞, taken to infinity, and define the correlator by a limiting procedure:[41]

$$\Psi_L(z_i)=\lim_{z_\infty\to\infty} z^{mN^2}\langle V_1(z_1)V_1(z_2)\dots V_1(z_{N-1})V_1(z_N)\rangle=\prod_{i<j}(z_i-z_j)^m\,. \tag{A1}$$

This prescription does not produce the exponential factor $e^{-\sum_i|z_i|^2/4\ell^2}$, characteristic of a lowest Landau level wave function.

In this paper we use the prescription given in Ref. 4, which corresponds to a smeared background charge given by the operator

$$e^{-i\sqrt{m}\rho_m\int d^2z\,\varphi(z)}\,, \tag{A2}$$

where $\rho_m=\rho_0/m$ with $\rho_0=eB/2\pi$ the density of filled Landau level. The difficulty with this prescription is that a direct evaluation of the correlator gives a contribution

$$e^{-m\rho_m\sum_i\int d^2z\,\ln(z-z_i)}\,, \tag{A3}$$

where the presence of the logarithm makes the imaginary part of the integral undefined. The aim of this appendix is to give a regularized version of the smeared background charge that: i) is well defined; ii) reproduces the pertinent gaussian factor in (8); and iii) differs from (8) only through a well defined (although singular) gauge transformation.

The idea behind our regularization is to replace the continuous background field by a lattice of singular flux tubes of strength

$$\delta\phi = \frac{\phi_0}{n} = \frac{2\pi}{ne}, \tag{A4}$$

which defines the integer n.

First consider a single electron in the presence of a single (fractional) flux tube at the origin. The wave function close to the flux tube behaves as

$$\psi(z) \sim z^{-\frac{\delta\phi}{2\pi}}, \tag{A5}$$

so for a single electron in a flux tube lattice, it is natural to consider a wave function of the type

$$\psi(z;\{z_{\vec{n}}\}) = f(z)\prod_{\vec{n}}(z-z_{\vec{n}})^{-\frac{\delta\phi}{2\pi}}, \tag{A6}$$

which has the correct singular behaviour at the lattice points $z_{\vec{n}}$ and satisfies the Laplace equation,

$$\nabla^2\psi(z;\{z_{\vec{n}}\}) = 4\partial_z\partial_{\bar{z}}\psi(z;\{z_{\vec{n}}\}) = 0 \quad ; \quad z \notin \{z_{\vec{n}}\}. \tag{A7}$$

This is not a sufficient condition for an acceptable electron wave function – we must also require that $\psi(z;\{z_{\vec{n}}\})$ is normalizable. As we now show, this will determine the allowed analytic functions $f(z)$. Without any loss of generality, we specialize to a regular lattice of K points $z_{\vec{n}} = (n_x + in_y)a$ with spacing a that covers a total area A - this is our regularized version of a uniform flux $\Phi = K\delta\phi$ corresponding to a uniform field of strength $B = \Phi/A$ inside the area A. We will ignore edge effects.

Since we have $\lim_{z\to\infty}\psi(z;\{z_{\vec{n}}\}) \sim f(z)z^{-K\frac{\delta\phi}{2\pi}}$, and normalizability requires[49] $\lim_{z\to\infty}\psi(z;\{z_{\vec{n}}\}) \sim 1/z$, we demand $f(z) \sim z^k$ where,

$$k - K\frac{\delta\phi}{2\pi} = k - \frac{\Phi}{2\pi} \leq -1, \tag{A8}$$

where we choose A so that $\Phi/2\pi$ is an integer. This can be rewritten as

$$k+1 \leq \frac{\Phi}{2\pi} = A\rho_0 = N_0, \tag{A9}$$

where N_0 is the number of states in the lowest Landau level. This result makes it plausible that, in the limit of large n, the functions (A6) with

$$f(z) = \sum_{k=0}^{N_0-1} c_k z^k$$

will give a good description of the lowest Landau level at a magnetic field of strength B. To show this, we rewrite ψ as

$$\psi(z;\{z_{\vec{n}}\}) = f(z)\prod_{\vec{n}}\left(\frac{z-z_{\vec{n}}}{\bar{z}-\bar{z}_{\vec{n}}}\right)^{-\frac{\delta\phi}{4\pi}}\prod_{\vec{n}}|z-z_{\vec{n}}|^{-\frac{\delta\phi}{2\pi}}, \tag{A10}$$

and approximate

$$\prod_{\vec{n}}|z-z_{\vec{n}}|^{-\frac{\delta\phi}{2\pi}} = \exp\left(-\frac{\delta\phi}{2\pi}\sum_{\vec{n}}\ln|z-z_{\vec{n}}|\right) \approx \exp\left(-\frac{\delta\phi}{2\pi}\frac{1}{a^2}\int d^2r'\,\ln|\vec{r}-\vec{r}'|\right) = e^{-\frac{\delta\phi}{4a^2}|z|^2}. \tag{A11}$$

The last integral was calculated as

$$\int d^2r'\,\ln|\vec{r}-\vec{r}'| = \frac{1}{4}\int d^2r'\,\ln|\vec{r}-\vec{r}'|\nabla^2_{r'}r'^2 = \frac{2\pi}{4}\int d^2r'\,\delta^2(\vec{r}-\vec{r}')r'^2 = \frac{\pi r^2}{2}, \tag{A12}$$

where we integrated by parts and neglected boundary terms. (The justification is that there is an understood density function $\rho(\vec{r}')$ that rapidly falls to zero outside the area A but is essentially constant inside. This still leaves an edge

correction due to the derivatives acting on the profile that we ignore.) Finally we note that $\frac{\delta\phi}{4a^2} = K\delta\phi\frac{1}{4Ka^2} = \Phi\frac{1}{4A} = \frac{eB}{4} = \frac{1}{4\ell^2}$, where ℓ is the magnetic length corresponding to the field strength B, so the approximate wave functions are of the form

$$\psi(z;\{z_{\vec{n}}\}) = \prod_{\vec{n}} \left(\frac{z-z_{\vec{n}}}{\bar{z}-\bar{z}_{\vec{n}}}\right)^{-\frac{\delta\phi}{4\pi}} f(z)e^{-\frac{1}{4\ell^2}|z|^2} \tag{A13}$$

and are expected to be valid in the limit of $a/\ell \to 0$, and z well inside the lattice. We have thus recovered the standard lowest Landau level wave functions, albeit in an unconventional gauge defined by the (arbitrarily chosen) flux lattice. This conclusion is also strongly suggested by the numerical calculations of Pryor, who shows that already for $n = 8$, at least four flat bands are identifiable in the electron spectrum, corresponding to the four lowest Landau levels[42]. A generalization of the analytic argument presented above to include higher Landau levels would be interesting.

Returning to our original goal of regularizing the operator insertion (A2), we see that

$$e^{-i\sqrt{m}\rho_m \int d^2z\,\varphi(z)} \to \prod_{\vec{n}} V_b(z_{\vec{n}}) = \prod_{\vec{n}} e^{-i\frac{a^2}{\sqrt{m}2\pi\ell^2}\phi(z_{\vec{n}})} \tag{A14}$$

will do the job in the limit $a/\ell^2 \to 0$, because the total $U(1)$ charge of the K vertex operators V_b equals $-Ka^2eB/2\pi m = -A\rho_m = -N$, where N is the total number of electrons. Making the replacement (A14) in (8) (A14) in (8)

$$\left\langle V_1(z_1)V_1(z_2)\ldots V_1(z_N)\prod_{\vec{n}} e^{-i\frac{a^2}{\sqrt{m}2\pi\ell^2}\phi(z_{\vec{n}})}\right\rangle, \tag{A15}$$

and using the same approximation as in (A11), we regain the correct exponential factor (The contraction of $V_1(z_j)$ and $\prod_{\vec{n}} V_b(z_{\vec{n}})$ gives the factor $\prod_{\vec{n}} |z_j - z_{\vec{n}}|^{-\delta\phi/2\pi}$ which, according to (A11), gives the Gaussian factor for z_j), an unimportant constant from the contractions between the different V_b's, and also a singular and rapidly changing phase factor just as in (A13). This is the regularized version of the statement in reference 4 that (8) "is trying to give us the answer in a gauge where the vector potential is zero, which means it differs by an everywhere-singular gauge transformation from the usual symmetric gauge vector potential for the uniform background magnetic field." Here we should also mention that in correlation functions involving the full scalar field, $\phi(z,\bar{z})$, the singular phases will cancel out, so these are well behaved functions even in the limit of vanishing lattice spacing. Finally, we note that the regularization procedure outlined above suffers from a formal difficulty - it introduces vertex operators with charges that do not belong to the charge lattice of the CFT under consideration. Putting a compensating charge at infinity would not suffer from this problem, but would also not correspond to a homogeneous system.

APPENDIX B: EQUIVALENCE BETWEEN CFT AND CF WAVE FUNCTIONS

In this appendix we provide derivations for some of the formulae in the main text, and prove that the CFT wave functions for the states in the Jain series indeed reproduce the wave functions from the CF framework using a direct projection on the LLL.

1. An identity

We begin by deriving the central relation (13). The basic idea is to express the antisymmetrization as a Slater determinant and then use the Laplace expansion of an $(N+M)\times(N+M)$ determinant,

$$\det A = \sum_i \epsilon_{\mathcal{P}_i} \det B_i \det C_i, \tag{B1}$$

where the sum is over the $\begin{pmatrix} M+N \\ N \end{pmatrix}$ ways in which a $N\times N$ matrix B_i can be formed from the first N rows of A, C_i is the complementary $M\times M$ matrix and ϵ_{P_i} is the sign of the permutation needed to bring the N columns of B_i followed by the M columns of C_i into the original order.[43] This formula generalizes the expansion by a row used to derive (11) for the one quasiparticle case.

We have,

$$\begin{aligned}
&\mathcal{A}\ \{\prod_{p<q}^{M}(z_p-z_q)^{1/m+l_{pq}}P_{\frac{1}{m}}(z_1)\dots P_{\frac{1}{m}}(z_M)V_1(z_{M+1})\dots\dots V_1(z_N)\} \qquad \text{(B2)}\\
&= \sum_{\mathcal{P}_r}\epsilon_{\mathcal{P}_r}\prod_{p<q}^{M}(z_{r_p}-z_{r_q})^{1/m+l_{pq}}P_{\frac{1}{m}}(z_{r_1})\dots P_{\frac{1}{m}}(z_{r_M})V_1(z_{r_{M+1}})\dots\dots V_1(z_{r_N})\\
&= \sum_{\{i_n\}}\epsilon_{\{i_n\}}\sum_{\mathcal{P}_s}\prod_{p<q}^{M}sgn_{\mathcal{P}_s}(z_{s_p}-z_{s_q})^{1/m+l_{pq}}P_{\frac{1}{m}}(z_{s_1})\dots P_{\frac{1}{m}}(z_{s_M})\sum_{\mathcal{P}_t}sgn_{\mathcal{P}_t}V_1(z_{t_{M+1}})\dots V_1(z_{t_N})\\
&= \sum_{\{i_n\}}(-1)^{\sum_{p=1}^{M}i_p}\mathcal{R}\{\prod_{p<q}^{M}(z_{i_p}-z_{i_q})^{1/m+l_{pq}}P_{\frac{1}{m}}(z_{i_1})\dots P_{\frac{1}{m}}(z_{i_M})V_1(z_{\bar{i}_{M+1}})\dots V_1(z_{\bar{i}_N})\}\ ,
\end{aligned}$$

where $\{r_1\dots r_N\}$ is a permutation, $\mathcal{P}_r$, of $\{1\dots N\}$; $\{s_1\dots s_M\}$ is a permutation, $\mathcal{P}_s$, of a subset $\{i_1\dots i_M\}$ of the N integers; $\{t_{M+1}\dots t_N\}$ a permutation, $\mathcal{P}_t$, of the conjugate set $\{\bar{i}_1\dots\bar{i}_M\}$ of $N-M$ integers; and $\epsilon_{\mathcal{P}_r}$ *etc.* the corresponding sign factors. The symbol l_{pq} denotes the relative angular momentum, and $\mathcal{R}$ is the radial ordering operator. The sum $\sum_{\{i_n\}}$ is over all the $N!/M!(N-M)!$ ways of doing this partition, and $\epsilon_{\{i_n\}}$ is the sign of the permutation needed to order the M rows of the first partition to the left in the Slater determinant. In the above expression, the second line is the definition, the third row follows from the Laplace expansion, and the last by noting that the prefactor makes the expression explicitly antisymmetric under exchange of coordinates in the subset $\{i_n\}$. Antisymmetry under exchange in the second subset is already guaranteed by the anti-commutation relations for the V_1:s. As in the single quasiparticle case, no extra signs are obtained by the final radial ordering because P and V_1 commute.

2. Equivalence between the $\nu = 2/5$ CF and CFT wave functions.

We now prove that the CFT wave function (22) for $\nu = 2/5$ is identical to that of composite fermions. Recalling that (22) differs from (18) only in that all the derivatives are on the left, we have the following explicit expression (recall that $N = 2M$ so there are M V_1:s and M V_2:s in the correlator),

$$\begin{aligned}
\Psi^{\rm CFT}_{2/5}(z_i) &= \sum_{i_1<i_2\dots i_M}(-1)^{\sum_k^M i_k}\partial_{z_{i_1}}\partial_{z_{i_2}}\dots\partial_{z_{i_M}}\prod_{k<l}^{M}(z_{i_k}-z_{i_l})^3 \qquad \text{(B3)}\\
&\prod_{k_1}^{(i_2,i_3\dots i_M)}(z_{k_1}-z_{i_1})^2\prod_{k_2}^{(i_1,i_3\dots i_M)}(z_{k_2}-z_{i_2})^2\dots\prod_{k_N}^{(i_1,i_2\dots i_M)}(z_{k_N}-z_{i_n})^2\prod_{m<n}^{(i_1,i_2\dots i_M)}(z_m-z_n)^3\,.
\end{aligned}$$

To write this in the CF form, we factor out a full Jastrow factor:

$$\Psi^{\rm CFT}_{2/5}(z_i) = \sum_{i_1<i_2\dots i_M}(-1)^{\sum_k^M i_k}\partial_{z_{i_1}}\partial_{z_{i_2}}\dots\partial_{z_{i_M}}\prod_{k<l}^{M}(z_{i_k}-z_{i_l})^1\prod_{m<n}^{(i_1,i_2\dots i_M)}(z_m-z_n)^1\prod_{p<q}^{N=2M}(z_p-z_q)^2 \qquad \text{(B4)}$$

The first two Jastow factors are nothing but the Vandermonde determinants of the subset $I = \{z_{i_1}\dots z_{i_M}\}$ and the conjugate subset $J = \{z_{\bar{i}_1}\dots z_{\bar{i}_M}\}$. Also useful is the operator identity

$$\partial_{z_1}\partial_{z_2}\dots\partial_{z_M}\begin{vmatrix}1 & 1 & . & . & 1\\ z_1 & z_2 & . & . & z_M\\ z_1^2 & z_2^2 & . & . & z_M^2\\ . & . & . & . & .\\ z_1^{M-1} & . & . & . & z_M^{M-1}\end{vmatrix} = \begin{vmatrix}\partial_{z_1} & \partial_{z_2} & . & . & \partial_{z_M}\\ \partial_{z_1}z_1 & \partial_{z_2}z_2 & . & . & \partial_{z_M}z_M\\ \partial_{z_1}z_1^2 & \partial_{z_2}z_2^2 & . & . & \partial_{z_M}z_M^2\\ . & . & . & . & .\\ \partial_{z_1}z_1^{M-1} & . & . & . & \partial_{z_M}z_M^{M-1}\end{vmatrix} \qquad \text{(B5)}$$

which follows because each coordinate, as well as the corresponding derivative, appears once and only once in every term when the determinant is expanded. We can now use the Laplace formula, (B1), in the opposite direction to

write

$$\Psi^{\rm CFT}_{2/5}(z_i) = \begin{vmatrix} \partial_{z_1} & \partial_{z_2} & . & . & \partial_{z_N} \\ \partial_{z_1} z_1 & \partial_{z_2} z_2 & . & . & \partial_{z_N} z_N \\ \partial_{z_1} z_1^2 & \partial_{z_2} z_2^2 & . & . & \partial_{z_N} z_N^2 \\ . & . & . & . & . \\ \partial_{z_1} z_1^{M-1} & . & . & . & \partial_{z_N} z_N^{M-1} \\ 1 & 1 & . & . & 1 \\ z_1 & z_2 & . & . & z_N \\ z_1^2 & z_2^2 & . & . & z_N^2 \\ . & . & . & . & . \\ z_1^{M-1} & . & . & . & z_N^{M-1} \end{vmatrix} \prod_{p<q}^{N}(z_p - z_q)^2 = \begin{vmatrix} \partial_{z_1} & \partial_{z_2} & . & . & \partial_{z_N} \\ z_1\partial_{z_1} & z_2\partial_{z_2} & . & . & z_N\partial_{z_N} \\ z_1^2\partial_{z_1} & z_2^2\partial_{z_2} & . & . & z_N^2\partial_{z_N} \\ . & . & . & . & . \\ z_1^{M-1}\partial_{z_1} & . & . & . & z_N^{M-1}\partial_{z_N} \\ 1 & 1 & . & . & 1 \\ z_1 & z_2 & . & . & z_N \\ z_1^2 & z_2^2 & . & . & z_N^2 \\ . & . & . & . & . \\ z_1^{M-1} & . & . & . & z_N^{M-1} \end{vmatrix} \prod_{p<q}^{N}(z_p - z_q)^2, \quad \text{(B6)}$$

where we omitted an unimportant sign. The last expression is, up to an overall normalization factor, the CF wave function as given in reference[36].

The last identity in (B6) follows because when the derivatives act on the factors in the determinant they give a row that is already present in the lower part of the determinant. For this to be true it is necessary that all the angular momentum states are present and that there are at least as many rows without derivatives as those with derivatives. These conditions correspond to having a maximum density droplet of electrons in the second CF Landau level which is no larger then the droplet formed by the electrons in the lowest CF Landau level. This completes the proof of the statement in the main text.

3. The general CF operators and the Jain series

We now extend the previous analysis to a general state in the Jain series. First we give the explicit expressions for the operators $V_{p,n}$ discussed in section III D:

$$\begin{aligned} V_{p,1}(z) &= e^{i\sqrt{2p+1}\varphi_1(z)} \\ V_{p,2}(z) &= \partial e^{i\frac{2p}{\sqrt{2p+1}}\varphi_1(z)} e^{i\sqrt{1+\frac{2p}{2p+1}}\varphi_2(z)} \\ V_{p,3}(z) &= \partial^2 e^{i\frac{2p}{\sqrt{2p+1}}\varphi_1(z)} e^{i\frac{2p}{\sqrt{(2p+1)(4p+1)}}\varphi_2(z)} e^{i\sqrt{1+\frac{2p}{4p+1}}\varphi_3(z)} \\ &\cdots \\ V_{p,n}(z) &= \partial^{n-1} e^{i\frac{2p}{\sqrt{2p+1}}\varphi_1(z)} e^{i\frac{2p}{\sqrt{(2p+1)(4p+1)}}\varphi_2(z)} \cdots e^{i\frac{2p}{\sqrt{[2p(n-2)+1][2p(n-1)+1]}}\varphi_{n-1}(z)} e^{i\sqrt{\frac{2np+1}{2(n-1)p+1}}\varphi_n(z)}. \end{aligned} \quad \text{(B7)}$$

Because all φ_i's commute, we can write

$$V_{p,n}(z) = \partial^{n-1} e^{i\tilde{\varphi}_n(z)}, \quad \text{(B8)}$$

where $\tilde{\varphi}_1 = \varphi_1$ and

$$\tilde{\varphi}_n(z) = \sum_{k=1}^{n-1} \frac{2p}{\sqrt{[2(k-1)p+1](2kp+1)}}\varphi_k(z) + \sqrt{\frac{2np+1}{2(n-1)p+1}}\varphi_n(z) \quad ; \quad n \geq 0 \quad \text{(B9)}$$

Using the sum formula:

$$\sum_{k=1}^{n} \frac{1}{[2p(k-1)+1][2pk+1]} = \frac{n}{2pn+1} \quad \text{(B10)}$$

and the charge density operator

$$J(z) = \frac{i}{\sqrt{2p+1}}\partial_z\varphi_1(z) + \frac{i}{\sqrt{(2p+1)(4p+1)}}\partial\varphi_2(z) \cdots + \frac{i}{\sqrt{[2p(n-1)+1][2pn+1]}}\partial\varphi_n(z) \quad \text{(B11)}$$

it can be shown that the operators (B7) satisfy the properties stated in the text vis á vis charge and statistics, and also give the filling fraction $\nu = \frac{n}{2pn+1}$. We can now construct the wave function for the general ground state in the

Jain series by a recursive procedure. For a total of $N = nM$ electrons, it is natural to write

$$\Psi^{CF}_{p,n}(z_i) \;=\; \mathcal{A}\{\langle \prod_{i=1}^{M} V_{p,n}(z_i) \prod_{j=M+1}^{2M} V_{p,n-1}(z_j)\cdots \prod_{j=(n-1)M+1}^{nM} V_{p,1}(z_j)\rangle\} \tag{B12}$$

The proof that this indeed reproduces the $\nu = n/(2pn+1)$ CF wave function, is a straightforward generalization of that given for 2/5 in the previous section. It involves using the Laplace formula (B1) iteratively $n-1$ times, breaking the problem down into the n groups (Landau levels) of particles, in analogy with the procedure in section B 1. The generalization of (B3) then contains $n-1$ sign factors, one for each additional group of particles, and one can follow the logic of (B4) - (B6) (with n $(M\times M)$ subdeterminants instead of two) to derive the equivalence of the CF and CFT wave functions.

APPENDIX C: THE NORMALIZATION FACTORS $\mathcal{N}_1$ AND $\mathcal{N}_2$

We begin with a single quasiparticle. Using the explicit form (44), and keeping only the diagonal terms in the double sum in the normalization integral we get,

$$|\tilde{\mathcal{N}_1}(\eta,\bar\eta)|^{-2} \sim \sum_i \int d^2z_i\, e^{-\frac{1}{2m}|z_i-\eta|^2} \int \prod_{j\neq i} d^2z_j\, \langle P_{\frac{1}{m}}(z_i)\prod_i V_1(z_i)\rangle\, \langle P_{\frac{1}{m}}(\bar z_i)\prod_i V_1(\overline{z}_i)\rangle^* . \tag{C1}$$

Here and below we use the sign $\sim$ to indicate that we neglect η-independent constants. We write $P_{\frac{1}{m}}(z_i) = \partial_i \hat P_{\frac{1}{m}}(z_i)$ and $P_{\frac{1}{m}}(\bar z_i) = \bar\partial_i \hat P_{\frac{1}{m}}(\bar z_i)$ and perform the z_i integral after making the approximate substitution $z_i \to \eta$ in the correlators. This gives

$$|\tilde{\mathcal{N}_1}(\eta,\bar\eta)|^{-2} \sim \sum_i \int \prod_{j\neq i} d^2z_j \left[\partial_\eta \langle \hat P_{\frac{1}{m}}(\eta)\prod_{j\neq i} V_1(z_j)\rangle\right]\left[\partial_{\bar\eta}\langle \hat P_{\frac{1}{m}}(\bar\eta)\prod_{j\neq i} V_1(\overline{z}_j)\rangle^*\right] \tag{C2}$$

Note that we first moved the derivatives in the operators $P_{\frac{1}{m}}(z)$ outside the expectation values. That this is allowed follows either from a direct calculation, or from noting that $P_{\frac{1}{m}}(z)$ is a descendant of the primary field $\hat P_{\frac{1}{m}}(z)$ and using standard methods to express the correlator of descendant fields as derivatives of correlators of primary fields[37]. It is important that all sign factors cancel in the diagonal terms.

Next we note that the $\bar\eta$ dependence of each correlator is given by $\langle \hat P_{\frac{1}{m}}(\eta)\prod_{j\neq i} V_1(z_j)\rangle \sim \exp\left[-(m-1)|\eta|^2/(4m)\right]$. This allows us to move the derivatives outside the full two dimensional correlation function. Reintroducing the magnetic length, ℓ, defining $D_\eta = \partial_\eta + c\,\bar\eta$ with $c = (m-1)/(m\ell^2)$, and noting $[D_\eta, \bar D_\eta] = 0$, we get,

$$|\tilde{\mathcal{N}_1}(\eta,\bar\eta)|^{-2} \sim \sum_i D_\eta \bar D_\eta \int \prod_{j\neq i} d^2z_j\, |\langle \hat P_{\frac{1}{m}}(\eta)\prod_{j\neq i} V_1(z_j)\rangle|^2 \tag{C3}$$

The right hand side of this equation is now in a form where plasma analogy arguments can be applied: the integral is the free energy of an overall neutral plasma with a charged impurity at the fixed postion η. This free energy is independent of the impurity positions because of screening, so finally, using $D_\eta \bar D_\eta 1 = c^2\,\bar\eta\eta + c$, and noting that all terms in the sum give identical contributions, we conclude that to leading order in $\ell^2/|\eta|^2$, $|\tilde{\mathcal{N}_1}(\eta,\bar\eta)|^{-2} \sim \bar\eta\eta$, which gives (53) in the main text.

The calculation of the two quasiparticle normalization factor, $|\tilde{\mathcal{N}_2}(N,\eta;\bar N,\bar\eta)|$ follows in an analogous manner, with some extra complication due to the more complicated exponential factors in the expression (45). Again keeping only the diagonal terms and completing squares in the exponents, we get,

$$\begin{aligned}|\tilde{\mathcal{N}_2}(N,\eta;\bar N,\bar\eta)|^{-2} \sim \frac{1}{\bar\eta\eta}\sum_{i<j}\int d^2z_{ij}d^2Z_{ij}\, e^{-\frac{1}{m}|Z_{ij}-N|^2}[e^{-\frac{1}{4m}|z_{ij}-\eta|^2} + e^{-\frac{1}{4m}|z_{ij}+\eta|^2} - 2\cos\vartheta\, e^{-\frac{1}{4m}(|\eta|^2+|z_{ij}|^2)}]\\ (z_{ij}\overline{z}_{ij})^{1-\frac{1}{m}}\int\prod_{k\neq i,j} d^2z_k\,[\partial_i\partial_j\langle \hat P_{\frac{1}{m}}(z_i)\hat P_{\frac{1}{m}}(z_j)\prod_{k\neq i,j} V_1(z_k)\rangle][\overline{\partial}_i\overline{\partial}_j\langle \hat P_{\frac{1}{m}}(\bar z_i)\hat P_{\frac{1}{m}}(\bar z_j)\prod_{k\neq i,j} V_1(\overline{z}_k)\rangle],\end{aligned} \tag{C4}$$

where $e^{i\vartheta} = \bar\eta z_{ij} - \overline{z}_{ij}\eta$, and overall constants are suppressed and the derivatives are moved outside the expectation values. Because of the gaussian factors in $|Z_{ij}-N|$, we can approximate the integral by substituting the maximum

value $Z_{ij} = N = \frac{1}{2}(\eta_+ + \eta_-)$. The third term in the square brackets ($\sim \cos\vartheta$) is maximum at $z_{ij} = 0$; the integral vanishes with this substitution because of the factor $(z_{ij}\overline{z}_{ij})^{1-\frac{1}{m}}$. This term is therefore neglected. The remaining two terms are equal. For the first term, we have $z_{ij} = \eta = \eta_+ - \eta_-$. Proceeding as before, defining $D_+ = \partial_{\eta_+} + c\,\bar{\eta}_+$ *etc*, and using that the $\bar{\eta}_+$ dependence of each correlator is $\sim \exp[-(m-1)|\eta|^2/(4m)]$, we get,

$$|\tilde{\mathcal{N}}_2(N,\eta;\bar{N},\bar{\eta})|^{-2} \sim \int \prod_{k\neq i,j} d^2z_k\, [\partial_i\partial_j \langle \hat{P}_{\frac{1}{m}}(z_i)\hat{P}_{\frac{1}{m}}(z_j) \prod_{k\neq i,j} V_1(z_k)\rangle][\bar{\partial}_i\bar{\partial}_j \langle \hat{P}_{\frac{1}{m}}(\bar{z}_i)\hat{P}_{\frac{1}{m}}(\bar{z}_j) \prod_{k\neq i,j} V_1(\bar{z}_k)\rangle] \tag{C5}$$
$$= D_+D_-\bar{D}_+\bar{D}_- \int \prod_{k\neq i,j} d^2z_k\, |\langle \hat{P}_{\frac{1}{m}}(z_i)\hat{P}_{\frac{1}{m}}(z_j) \prod_{k\neq i,j} V_1(z_k)\rangle|^2.$$

The integral is now the partition function for a neutral plasma with two impurities at positions η_+ and η_- and this free energy is again independent of the impurity positions because of screening. We thus have $D_+D_-\bar{D}_+\bar{D}_-1 = c^2[c^2|\eta_+|^2|\eta_-|^2 + c|\eta_+|^2 + c|\eta_-|^2 + 1]$. Finally, taking $N = 0$ and substituting $z_\pm = \pm\eta/2$ in the above expressions, and noting that all terms in the sums give identical contributions, we get, to leading order in $\ell^2/|\eta|^2$, the formula (54) quoted in the text. Retaining the leading order contribution is valid in the limit when the quasiparticles are far separated; this suggests corrections to statistics for smaller separations, as also found in direct numerical evaluations[15,16].

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[44] Later the non-Abelian statistics has also been understood in the context of d-wave paired superconductors[8] and has also been studied numerically[9].
[45] An asymmetry in the description of the quasiparticles and quasiholes at $\nu = 1/m$ has been a striking feature of other descriptions as well. In Laughlin's theory, a quasihole at position η is represented as a vortex $\prod_i (z_i - \eta)$, while a quasiparticle is created by a complicated operator involving many derivatives. The fractional statistics of the quasiholes is easy to derive, while the statistics of the quasiparticles eludes a precise analytical treatment. In the Ginzburg-Landau-Chern-Simons effective theories, the quasiholes and quasiparticles are described by vortices and anti-vortices respectively, and again there is an asymmetry in the description.[11]
[46] A more careful evaluation of the correlators using a regularized background charge (*cf.* Appendix A) does give a contribution also from the exponential. In order to have holomorphic wave function this must be cancelled by replacing ∂_i with a suitable covariant derivative. Since this in the end amounts to a mere change of notation, we simply use the rule that the derivatives do not act on the exponential part in the correlation function.
[47] As straightforward projection tends to get computationally heavy in numerical calculations with many particles and a large number of derivatives, slightly different methods of obtaining LLL wave functions have been employed in most of the CF literature[13]. These, too, are often referred to as projection. For a single quasiparticle the different prescriptions agree, while in the general case they produce very similar but not identical wave functions. It is the brute force projection which exactly matches with the CFT construction for the Jain sequence ground states.
[48] The quantity $\mathcal{N}(\bar{\eta}\eta)$ can be interpreted as the (diagonal element) of the quasihole density matrix, $\mathcal{N}(\eta, \bar{\eta})^2 \equiv \rho(\eta, \bar{\eta}) = \psi(\eta)\psi(\bar{\eta})^\star$, where the factor $\psi(\eta_1, \eta_2)$ is the quasihole wave function.
[49] This is the correct condition on a sphere (or a compactified plane). On a real plane the last mode will be marginal in the sense that the normalization integral will have a logarithmic divergence.